

The University of Chicago

THE CHARACTERISTIC VALUE
PROBLEM OF HERMITIAN FUNCTION-
AL OPERATORS IN A NON-
HILBERTIAN SPACE

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
PHYSICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

MARCH, 1933

By
YUAN-YUNG TSENG

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INTRODUCTION

Broadly speaking, the characteristic value problem seeks to determine for a given operator all the numbers and operands which are more or less characteristic of the initial operator, or those operands of such nature that through the operator they are transformed into some exact or approximate multiples of themselves having the numbers in question as proportional factors. The operator, if we prefer concrete examples, may be a matrix operating on vectors as in algebra, a transformation on points as in geometry, a kernel on functions as in the theory of integral equations, or an observable of states as in Dirac's theory of quantum mechanics. In these widely different fields, it turns out that, among the operators which have played a prominent role of growing significance are those of the hermitian type--a type that enjoys certain property of symmetry. Indeed, since 1926, a very frequent, and fundamentally, perhaps a most important, mathematical problem in the new mechanics has been that of characteristic values for a hermitian operator. The solution of this problem is the principal objective of the present study.

Although solutions to the algebraic and geometric cases of the problem were discovered long ago by Gauss, Hermite, Pincherle and others, the transcendental problem came to be systematically investigated much later. It was Hilbert who, in 1906, first introduced a space of infinitely many dimensions that now bears his name, cast the problem into a suitable invariantive form, and solved it for bounded quadratic forms of infinitely many variables. Then, Hellinger, Riesz, Schmidt, Toeplitz and others of the Hilbert

school made further contributions to perfect the theory. The year 1923 saw a major advance by Carleman in whose work the boundedness restriction was entirely removed. In place of bounded kernels he used kernels of class I. To lay the requisite mathematical foundation of the quantum theory von Neumann set up, significantly enough, an abstract Hilbert space, and explicitly reformulated the problem without any reference to boundedness. This series of continual efforts culminated in 1929, when von Neumann, and Stone independently published final results which form the complete solution of the problem for hypermaximal or self-adjoint operators in complex Hilbert space. Distinctly another order of generalizations is found in the second¹ theory of general analysis that E. H. Moore developed since 1915. According to Moore, not only the denumerable identity matrix but each and every positive hermitian matrix on an unconditioned double range gives rise to modular functions, modular matrices, and he showed that the problem admits solution for any modular hermitian matrix.

These studies, comprehensive as they are, do not, however, seem to have exhausted the possibilities. They show one interesting divergence: in general analysis the matrices considered for this problem are always modular, whereas the recent theory of operators invariably confines itself to spaces with a denumerable infinitude of dimensions. It is noteworthy that F. Riesz in his paper of 1930 pointed out (without proof) certain consequence of the orthogonal decomposition of space as still valid, regardless of the separability condition. But the representation theorem

¹Moore's earlier generalization of the Hilbert-Schmidt theory of integral equations will be discussed elsewhere.

of linear functionals was explicitly taken to be dependent on that condition. In particular, Dirac, in his theoretical works has repeatedly made use of abstract spaces with no dimension restriction of any kind.¹ We are thus in need of a more general solution of the problem.

In the present paper,² neither is the space necessarily separable, nor for our problem do we assume the operators bounded. More specifically, we propose for study a space³ $\mathcal{M}$ which includes Hilbert space, Moore's modular space and others as special instances. It is characterized abstractly by three internal properties:

(a) It is linear (real, complex, or quaternionic).

(b) For any pair of points (vectors, or functions) there is defined a numerical inner product.

(c) Every convergent Cauchy sequence has a limiting point in the space.

As to operators we consider almost exclusively point transformations Ω carrying points of a linear set dense⁴ on $\mathcal{M}$ into points of $\mathcal{M}$ with

$$(\Omega\xi, \mu) = (\xi, \Omega\mu),$$

¹See, for instance, Dirac, The principles of quantum mechanics, 1930, esp. pp. 63 ff., where, as usual, a number of mathematical questions of existence and convergence remain to be settled.

²Here most of the results and proofs are given still in the form as they were previously obtained by the author in 1930-1931.

³An example of this space can be given such that it is neither a Hilbert space nor a modular space.

⁴A linear subspace is dense on the complete main space if the main space has no vector, different from zero, that is orthogonal to the subspace.

where parentheses denote the inner products, ξ, μ any points in the definition range of Ω , and $\Omega\xi$ stands for the resultant point obtained by operating on ξ by Ω . Such operators are said to be hermitian. The characteristic value problem then runs as follows:

Given a hermitian operator Ω , to find a one-parameter family E_t , t varying from $-\infty$ to $+\infty$, of hermitian operators such that

- (1) $E_s E_t = E_t E_s = E_s$ for $s \leq t$,
- (2) $E_t \rightarrow Z$ as $t \rightarrow -\infty$, $E_t \rightarrow I$ as $t \rightarrow +\infty$,
- (3) $E_{t-e} \rightarrow E_t$ as $e \rightarrow 0$, $e > 0$,
- (4) $\Omega\xi = \int_{-\infty}^{+\infty} t dE_t\xi$,

Z, I being the zero and identity operators, ξ a point in the definition range of Ω .

As it stands, the problem may not admit any solution. We determine a necessary and sufficient condition for its solvability and the solution, when existent, is effected by stages.

Chapter I introduces an abstract space with its defining properties. Four alternatives to the postulate of completeness are studied. Then we establish some fundamental theorems on general Cauchy conditions, orthogonal decomposition of vectors, and the representation of linear functionals. It is shown that every system of functions (not necessarily orthogonal nor supposed to be denumerable) gives rise to a Fourier situation. A functional integral takes the place of the Fourier series. Some necessary, and sufficient conditions for the existence of this integral are found. In a space not known to be complete, the Bessel inequality and a formula for a best approximation are obtained. Moreover, we find the Fourier and Parseval equalities always equivalent.

If the space is complete, then, by means of the Fourier coefficients, every function can be expanded according to the initial system, provided the function lies in a certain subspace closely connected with that system. Geometrically, every Fourier transformation is a projection. Two well-known theorems receive also complete generalizations here: the Parseval equality and the Riesz-Fischer theorem.

The study of operators begins in Chapter II. We give a number of equalities and inequalities concerning the moduli of operators. The idea of a numerical function of limited variation is first extended to an operatorial-valued function where the operators may be non-linear. Then we introduce operatorial measures of point-sets, and establish their requisite properties. As projection operators play such an important part in the modern operator theory, these operators are treated at some length. There is a one-to-one correspondence between projections and complete unitary subspaces. Finally, it is found that certain functional Hellinger integrals have a geometric significance: they are projection operators.

In Chapter III we first define operatorial correspondents for functions of the first order. With the aid of some of these functions a solution to the problem is then obtained in the case of a bounded operator. In terms of this solution every operatorial function can be expressed as a Stieltjes integral whose integrand is the initial numerical function. The integral representation in turn enables us to define and study operatorial correspondents to bounded discontinuous functions of any Baire class. Besides, we introduce another type of correspondence such that functions of a singular matrix, for example, remain singular. A simple

equation, however, connects these two types of operatorial functions.

Results contained in the previous two chapters are mostly preparatory for our study of the problem. Chapter IV solves the problem and develops a theory of characteristic values. Self-adjoint operators are defined and properties of their resolvents discussed. To every finite real number there corresponds, relative to the given self-adjoint operator, a projection operator separating the initial operator into two orthogonal semi-bounded parts. This separation operator is uniquely determined by its permutability character and its close relation with the characteristic values. As the real number varies the separation operator becomes a monotone increasing function of a real variable. Conversely, for every one-parameter family of separations which is monotone increasing there exists one and only one self-adjoint operator having these as its separations. By means of the separations we show that the problem has a solution if and only if the given operator is self-adjoint. There can be only one solution, and the solution is precisely the family of separations. Further, the solution is a function of limited variation in the operatorial sense, whose points of discontinuity give all the characteristic values and no others. The number of discontinuities may be none, finite, denumerable, or non-denumerable. The spectrum of a self-adjoint operator, its continuous and discontinuous spectra are introduced analytically and later characterized geometrically. The spectrum is a non-empty closed set whose left and right extremities coincide with the lower and upper moduli of the given operator. Instead of the familiar decomposition of a solution into continuous and discontinuous parts we make a three-part

decomposition: the first part is a discontinuous jump function, the second a barely continuous function with its derivative almost everywhere zero, and the third is absolutely continuous, equal to the indefinite Lebesgue integral of its derivative, all being operatorial functions of limited variation.

The last chapter is devoted to expansions with respect to the characteristic functions and characteristic differential solutions of a given self-adjoint operator. Under mild conditions certain functional Stieltjes, Lebesgue-Stieltjes, and Hellinger integrals are identified, and shown to converge strongly. To every self-adjoint operator there exist three complete orthonormal systems of characteristic functions and characteristic differential solutions such that the first system consists of characteristic functions, each characteristic differential solution in the second is barely continuous with a derivative vanishing almost everywhere, and that of the third is absolutely continuous, equal to the integral of its derivative. In terms of these, the initial operator and the identity operator can be expanded into strongly convergent general series by means of strict functional Hellinger integrals over the entire infinite interval, which exist as strong limits. Similar expansions are also given for Borel-measurable functions (bounded or unbounded) of a self-adjoint operator.

As most of the second theory of general analysis has been available only in lecture notes, in the appendix we state for the reader's reference some results from that theory which find applications here.

In the course of this study the works listed below have proved particularly useful:

- Carleman, T. Sur les equations integrales singulieres a noyau reel et symetrique. Uppsala, 1923.
- Hellinger, E. "Neue Begründung der Theorie quadratischer Formen von unendlichvielen Verändlichen." Journal für Math. 136 (1906), pp. 210-71.
- Hellinger, E. - Toeplitz, O. (1) "Grundlagen für eine Theorie der unendlichen Matrizen." Math. Ann. 69 (1910), pp. 289-330.
- (2) "Integralgleichungen und Gleichungen mit unendlichvielen Unbekannten." Encyklopadie der Math. Wiss. II C 13 (1928), pp. 1335-1597.
- Hilbert, D. "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen." Erste Mitt., Vierte Mitt. Göttinger Nachr. (1904), pp. 49-91; (1906), pp. 157-227.
- Moore, E. H. "Hermitian matrices of positive type in general analysis." Lectures at the University of Chicago.
- Neumann, J. v. (1) "Mathematische Begründung der Quantenmechanik." Göttinger Nachr. (1927), pp. 1-57.
- (2) "Allgemeine Eigenwerttheorie hermitescher Funktionaloperatoren." Math. Ann. 102 (1929), pp. 49-131.
- Riesz, F. (1) Les systemes d'equations lineaires a une infinite d'inconnues. Paris, 1913.
- (2) "Über die linearen Transformationen des komplexen Hilbertschen Raumes." Szeged Acta 5 (1930), pp. 23-54.
- Schmidt, E. (1) "Entwicklung willkürlicher Funktionen nach Systemen vorgeschrieben." Math. Ann. 63 (1907), pp. 433-76.
- (2) "Über die Auflösung linearer Gleichungen mit unendlichvielen Unbekannten." Palermo Rendiconti 25 (1908), pp. 53-77.
- Stone, M. H. "Linear transformations in Hilbert space." I, II, III. Proc. Nat. Acad. U.S.A. 15 (1929), pp. 198-200, pp. 423-5; 16 (1930), pp. 172-5.
- Weyl, H. (1) "Singular Integralgleichungen." Math. Ann. 66 (1908), pp. 273-324.
- (2) "Über beschränkte quadratische Formen, deren Differenz vollstetig ist." Palermo Rendiconti 27 (1909), pp. 373-92.
- Wintner, A. Spektraltheorie der unendlichen Matrizen. Leipzig, 1929.

The author is especially indebted to the investigations of Moore and of Riesz whose methods are utilized in the following pages.

Obviously the introduction of this space raises many problems for serious study. However, we may note that since the present problem has already been solved in this paper it involves no unusual difficulty to treat normal operators and unbounded functions of permutable self-adjoint operators. A solution theory for self-adjoint functional equations can be readily developed by direct application of the results of Chapter IV. Integral equations with hermitian kernels in the first and second theories of general analysis have closer connections, when viewed from a more general standpoint than that taken here. These related problems the author has approximately ready for publication.

CHAPTER I

UNITARY SPACES AND THE FOURIER SITUATION

1. Characterization of the space

In connection with linear equations various spaces have been proposed for study. As those of special interest we may mention the euclidean space of finite dimensions, Hilbert space, Hellinger integrable functions, and quadratically Lebesgue-summable functions. All these are unitary spaces. What is a unitary space? Let us consider (1) the set $\mathcal{O}$ of all real numbers, or of all complex numbers, or of all real quaternions, (2) a set $\mathcal{M}$ of elements (points, vectors, functions) μ , ξ , ζ , ..., and (3) functions x on $\mathcal{M}$ to $\mathcal{M}$, x on $\mathcal{O}$ to $\mathcal{M}$, and $(\cdot, \cdot)$ on $\mathcal{M}$ to $\mathcal{O}$. If the system $(\mathcal{O}, \mathcal{M}, x, (\cdot, \cdot))$ satisfies the first nine conditions of the following list then $\mathcal{M}$ is called a unitary space; if in addition the tenth is also satisfied we call $\mathcal{M}$ a complete unitary space. It should be noted that a complete unitary space is non-hilbertian in that such a space can have non-denumerably many dimensions. We use here the term unitary space tentatively. It has been applied almost exclusively to spaces of a finite number of dimensions satisfying also the next ten conditions.

A. Linearity

$$1. (\mu_1 + \mu_2) + \mu_3 = \mu_1 + (\mu_2 + \mu_3)$$

$$2. a_1(a_2\mu) = (a_1a_2)\mu$$

$$3. 1\mu = \mu$$

¹We speak of a function as on the independent range to the dependent range.

$$4. a(\mu_1 + \mu_2) = a\mu_1 + a\mu_2$$

$$5. (a_1 + a_2)\mu = a_1\mu + a_2\mu$$

B. Existence of a positive hermitian bilinear inner product (,)

6. The function (,) is bilinear.

$$(\mu_1 + \mu_2, \mu) = (\mu_1, \mu) + (\mu_2, \mu) \quad , \quad (a\mu_1, \mu_2) = a(\mu_1, \mu_2) .$$

7. (,) is positive. $(\mu, \mu) \geq 0$.

8. (,) is hermitian. $(\mu_1, \mu_2) = (\mu_2, \mu_1)$.

Definition.

$$-\mu \equiv (-1)\mu \quad , \quad \mu_1 - \mu_2 \equiv \mu_1 + (-\mu_2) \quad , \quad \|\mu\| \equiv (\mu, \mu)^{1/2}$$

$$9. \quad \|\mu_1\| = 0 = \|\mu_2\| \quad \text{implies} \quad \mu_1 = \mu_2 .$$

Definition. Let $\mathcal{L}$ be a set of elements χ among which holds a binary transitive and compositive relation¹ R. Then a system $\{\mu_\chi\}$ is said to converge strongly to a μ relative to R in case

$$\lim_{\chi \in \mathcal{L}} \|\mu_\chi - \mu\| = 0 ,$$

and we write $\lim_{\chi \in \mathcal{L}} \mu_\chi = \mu$ or simply $\lim_{\chi \in \mathcal{L}} \mu_\chi = \mu$.

C. Completeness

10. For every sequence $\{\mu_m\}$ with $\lim_{m,n} \|\mu_m - \mu_n\| = 0$ there exists a μ such that $\lim_{m \rightarrow \infty} \mu_m = \mu$.

We see readily that the space of almost periodic functions, the set $\mathcal{M}$ of basis Σ_5 in the first theory of E. H. Moore's general analysis,² and his modular functions³ are all unitary. A new example of complete linear unitary space may be

¹See Appendix

² Proc. Fifth International Congress of Mathematicians, Cambridge, 1912, Vol. I, pp. 232 ff.

³In the sequel this theory of modular functions will be referred to as the *MPF*-theory. See Appendix.

constructed as follows:

Let $\mathcal{P}$ be an unrestricted set of elements p . Consider a real single-valued function β on $\mathcal{P}$. Introduce the matrix $\epsilon_{\beta\sigma}(p_1 p_2) \equiv (1/2) |\beta(p_1) + \beta(p_2)| - (1/2) |\beta(p_1) - \beta(p_2)| - \underline{B} \beta(\sigma)$ for every finite subset σ of $\mathcal{P}$ and a pair $p_1 p_2$ in σ . If μ is a function on $\mathcal{P}$ to $\mathcal{A}$, define $J_{\beta\sigma} \mu \bar{\mu} \equiv \sum_{\sigma} \sum_{\sigma} \mu \gamma_{\beta\sigma} \bar{\mu}$ where $\gamma_{\beta\sigma}$ is the general reciprocal¹ of $\epsilon_{\beta\sigma}$. If μ is such that $\mu(p_1) = \mu(p_2)$ whenever $\beta(p_1) = \beta(p_2)$, then the least upper bound of $J_{\beta\sigma} \mu \bar{\mu}$ is said to be the modulus of μ , denoted by $M\mu$, and in case $M\mu$ is finite we call μ modular. The limit $\underline{L}_{\sigma} J_{\beta\sigma} \mu_1 \bar{\mu}_2$ as σ swells defines a bilinear functional $J\mu_1 \bar{\mu}_2$ when μ_1, μ_2 are both modular.

That these modular functions form a linear unitary space should be clear. To show its completeness we start with the hypothesis that $\underline{L}_{mn} (M(\mu_m - \mu_n)) = 0$. Hence $\underline{L}_m M\mu_m$ exists. For every p and σ containing p we have then

$$\begin{aligned} |\mu_m(p) - \mu_n(p)| &= |J_{\sigma} \epsilon_{\sigma}(p) (\mu_m - \mu_n)| \leq (\epsilon_{\sigma}(p))^{1/2} M_{\sigma} (\mu_m - \mu_n) \\ &\leq (\epsilon_{\sigma}(pp))^{1/2} M(\mu_m - \mu_n). \end{aligned}$$

Thus $\underline{L}_{mn} |\mu_m(p) - \mu_n(p)| = 0$ for every p and $\underline{L}_m \mu_m$ exists. This limit we denote by μ^* . So, $\underline{L}_m M_{\sigma} (\mu_m(\sigma) - \mu^*(\sigma)) = 0$.

$$M_{\sigma} \mu^* \leq \underline{B}_{\sigma} \underline{B}_m M_{\sigma} \mu_m = \underline{B}_m \underline{B}_{\sigma} M_{\sigma} \mu_m = \underline{B}_m M \mu_m < \infty,$$

since $M_{\sigma} \mu^* = \underline{L}_m M_{\sigma} \mu_m$ and $\underline{L}_m M \mu_m$ exists. Thus for the upper bound we have $\underline{B}_{\sigma} M_{\sigma} \mu^* \leq \underline{B}_m M \mu_m < \infty$ so that μ^* is modular. By hypothesis, there is an m_e such that $M(\mu_m - \mu_n) \leq e$ for every positive e and every pair $m > m_e, n > m_e$. In particular,

¹Ibid.

$M_\sigma(\mu_m - \mu_n) \leq e$. We have then $\lim_{n \rightarrow \infty} M_\sigma(\mu_m - \mu_n) = M_\sigma(\mu_m - \mu^*)$. As $M_\sigma \mu$ is monotone increasing in σ , $\int_\sigma M_\sigma \mu = \lim_{\sigma \rightarrow \infty} M_\sigma \mu$, and consequently $\int_\sigma M_\sigma(\mu_m - \mu^*) = M(\mu_m - \mu^*) \leq e$ for every positive e . Therefore $\lim_{m \rightarrow \infty} M(\mu_m - \mu^*) = 0$.

Remarks. In a complete linear metric space $\mathcal{M}$ every element ξ has a corresponding bounded linear functional L_ξ on $\mathcal{M}$ to α such that $L_\xi(\xi) = \|\xi\|$. Suppose that this L_ξ is unique for each ξ , $\varphi_{\xi+\mu} = \varphi_\xi + \varphi_\mu$, $\varphi_{a\xi} = \varphi_\xi \cdot a$, where $\varphi_\xi \equiv \|\xi\| L_\xi$. Then $\mathcal{M}$ is a unitary space. Or, in a linear metric space $\mathcal{M}$ simply assume (1) that to every ξ there exists one and only one bounded linear functional φ_ξ with $\varphi_\xi(\xi) = \|\xi\|^2$, (2) $\varphi_{\xi+\mu} = \varphi_\xi + \varphi_\mu$, (3) $\varphi_{a\xi} = \varphi_\xi \bar{a}$. It can be shown without difficulty that under these conditions $\mathcal{M}$ is unitary with the inner product (ξ, μ) equal to $(1/2) [\varphi_\xi(\mu) + \varphi_\mu(\xi)]$ for real numbers, to $\varphi_\mu(\xi)$ for complex or quaternionic numbers. However, it is not always necessary to bring into consideration the corresponding linear functionals. Thus, for Dirac's system of states $\{\psi\}$ we may introduce $(\psi_1, \psi_2) = \psi_1 \psi_2$, where ψ_2 is the "conjugate" of ψ_1 , so that this system is also unitary.

2. Fundamental properties

For a linear unitary space we have the results stated in Theorems 1-3.

THEOREM 1.

1. There exists one and only one point 0 such that $\|0\| = 0$.
2. For every point μ it is true that

$$0 \mu = 0, \quad \mu + 0 = \mu = 0 + \mu, \quad -(-\mu) = \mu, \\ \mu - \mu = 0 = -\mu + \mu.$$

Also if $\mu_1 - \mu_2 = 0$ then $\mu_1 = \mu_2$.

$$3. (\mu, \mu_1 + \mu_2) = (\mu, \mu_1) + (\mu, \mu_2) \quad , \quad (\mu_1, a\mu_2) = (\mu_1, \mu_2) \overline{a}.$$

4. For every pair of points μ_1, μ_2 it is true that
 $\mu_1 + \mu_2 = \mu_2 + \mu_1.$

$$5. \|a\mu\| = |a| \cdot \|\mu\|.$$

6. Let a_m, b_n be two sets of numbers, and $\mu_m, \{n$ be sets
of vectors, $m = 1 \dots r, n = 1 \dots s$. Then the next equality
holds.

$$\left(\sum_{m=1}^r a_m \mu_m, \sum_{n=1}^s b_n \{n \right) = \sum_{m=1}^r \sum_{n=1}^s a_m (\mu_m, \{n) \overline{b_n}.$$

Here (1), (2), (3), (5) are immediate consequences. To show (4) we note that¹

$$\begin{aligned} \mu_1 + \mu_2 / 2 &= (2 - 1)(\mu_1 + \mu_2 / 2) && \text{(by A (3))} \\ &= 2(\mu_1 + \mu_2 / 2) - (\mu_1 + \mu_2 / 2) && \text{(by A (4))} \\ &= 2\mu_1 + \mu_2 - \mu_1 - \mu_2 / 2 && \text{(by A (4); A (2),} \\ &&& \text{A (3), A (1), (2)).} \end{aligned}$$

Adding $\mu_2 / 2$ to both sides and using A (1), A (5), A (3), (2),

$$\mu_1 + \mu_2 = 2\mu_1 + \mu_2 - \mu_1.$$

Adding $-\mu_1$ to both sides and using A (1), (2), A (5), (2), A (3),

$$\mu_2 = \mu_1 + \mu_2 - \mu_1.$$

Hence adding μ_1 we have by A (1), (2),

$$\mu_2 + \mu_1 = \mu_1 + \mu_2$$

(6) is a result of direct calculation.

¹This proof is due to L. M. Graves.

Definition. We say $\{\mu_\ell\}$ converges weakly to μ_* in case $\sum \|\mu_\ell\| < \infty$ ¹ and $\sum (\mu, \mu_\ell) = (\mu, \mu_*)$ for every μ . This fact is expressed by $\sum \mu_\ell = \mu_*$.

THEOREM 2.

1. The gramian matrix $((\mu_{p_1}, \mu_{p_2}))$ for $p_1 p_2$ varying over an arbitrary class $\mathcal{P}$, is positive hermitian.²

$$2. |(\mu_1, \mu_2)| \leq \|\mu_1\| \cdot \|\mu_2\|$$

$$3. \left| \|\mu_1\| - \|\mu_2\| \right| \leq \|\mu_1 \pm \mu_2\| \leq \|\mu_1\| + \|\mu_2\|$$

(1) It is known that a matrix is positive hermitian if and only if the corresponding hermitian quadratic form is positive. Consequently the gramian matrix is positive hermitian, since

$$\sum_{\sigma} \sum_{\tau} a_{\sigma} (\mu_{p_1}, \mu_{p_2}) \overline{a_{\tau}} = \left(\sum_{\sigma} a_{\sigma} \mu_{p_1}, \sum_{\tau} a_{\tau} \mu_{p_2} \right)$$

σ being a finite set of the p 's.

(2) is a special case of (1).

For (3) we notice

$$|\text{real part of } (\mu_1, \mu_2)| \leq |(\mu_1, \mu_2)| \leq \|\mu_1\| \cdot \|\mu_2\|$$

$$-\|\mu_1\| \cdot \|\mu_2\| \leq \pm \text{real part of } (\mu_1, \mu_2)$$

$$\left| \|\mu_1\| - \|\mu_2\| \right|^2 = \|\mu_1\|^2 + \|\mu_2\|^2 - 2\|\mu_1\| \cdot \|\mu_2\|$$

$$\leq \|\mu_1\|^2 + \|\mu_2\|^2 \pm 2 \text{ r.p. } (\mu_1, \mu_2) = \|\mu_1 \pm \mu_2\|^2$$

$$\|\mu_1 \pm \mu_2\|^2 \leq \|\mu_1\|^2 + \|\mu_2\|^2 + 2 |\text{r.p. } (\mu_1, \mu_2)|$$

$$\leq \|\mu_1\|^2 + \|\mu_2\|^2 + 2 \|\mu_1\| \cdot \|\mu_2\| = (\|\mu_1\| + \|\mu_2\|)^2$$

¹For a denumerable sequence this condition becomes redundant.

²A hermitian matrix is said to be positive, if its principal minor determinants are positive. See Appendix.

THEOREM 3.

1. If $\underline{L}_\chi \mu_\chi = \mu_1$ and $\underline{L}_\chi \mu_\chi = \mu_2$ then $\mu_1 = \mu_2$.
2. In order that $\underline{L}_\chi \mu_\chi = \mu^*$ it is necessary and sufficient that $L_\chi(\mu_\chi, \mu) = (\mu^*, \mu)$ for every μ and $L_\chi \|\mu_\chi\| = \|\mu^*\|$. Further, if $\underline{L}_\chi \mu_\chi = \mu^*$ then $\underline{L}_\chi \mu_\chi = \mu^*$.
3. If $\underline{L}_{\chi_1} \mu_{\chi_1} = \mu_1$ and $\underline{L}_{\chi''} \mu_{\chi''} = \mu_2$, then

$$L_{\chi_1 \chi''}(\mu_{\chi_1}, \mu_{\chi''}) = (\mu_1, \mu_2) = L_{\chi_1} L_{\chi''}(\mu_{\chi_1}, \mu_{\chi''}) = L_{\chi''} L_{\chi_1}(\mu_{\chi_1}, \mu_{\chi''})$$

4. If $\underline{L}_\chi \mu_\chi = \mu^*$, and if $\underline{L}_\chi \xi_\chi = \xi^*$, then

$$L_{\chi_1 \chi_2}(\mu_{\chi_1}, \xi_{\chi_2}) = (\mu^*, \xi^*) = L_\chi(\mu_{\chi_1}, \xi_{\chi_2}).$$

(1) is an easy consequence of Theorem 1.

(2) The set of conditions is necessary.

$$(\mu_\chi, \mu) - (\mu^*, \mu) = (\mu_\chi - \mu^*, \mu)$$

$$\therefore |(\mu_\chi, \mu) - (\mu^*, \mu)| \leq \|\mu_\chi - \mu^*\| \cdot \|\mu\|.$$

As $L_\chi \|\mu_\chi - \mu^*\| = 0$, $L_\chi |(\mu_\chi, \mu) - (\mu^*, \mu)| = 0$, i.e., $L_\chi(\mu_\chi, \mu) = (\mu^*, \mu)$. Also $|\|\mu_\chi\| - \|\mu^*\|| \leq \|\mu_\chi - \mu^*\|$. Hence $L_\chi \|\mu_\chi\| = \|\mu^*\|$. The set is also sufficient. $\|\mu_\chi - \mu^*\|^2 = \|\mu_\chi\|^2 + \|\mu^*\|^2 - (\mu_\chi, \mu^*) - (\mu^*, \mu_\chi)$. Now $L_\chi \|\mu_\chi\| = \|\mu^*\|$, $L_\chi(\mu_\chi, \mu^*) = (\mu^*, \mu^*)$. Hence $L_\chi \|\mu_\chi - \mu^*\| = 2\|\mu^*\| - (\|\mu^*\| + \|\mu^*\|) = 0$. The last part becomes obvious when we recall that if $L_\chi \|\mu_\chi\| = \|\mu\|$ then $L_\chi \|\mu_\chi\| < \infty$.

$$(3) (\mu_{\chi'}, \mu_{\chi''}) - (\mu_1, \mu_2) = (\mu_{\chi'}, \mu_{\chi''} - \mu_2) + (\mu_{\chi'} - \mu_1, \mu_2)$$

$$\therefore |(\mu_{\chi'}, \mu_{\chi''}) - (\mu_1, \mu_2)| \leq \|\mu_{\chi'}\| \cdot \|\mu_{\chi''} - \mu_2\| + |(\mu_{\chi'} - \mu_1, \mu_2)|$$

As $L_{\chi'} \|\mu_{\chi'}\| < \infty$, there exist an χ_0' and a number b such that $\|\mu_{\chi'}\| \leq b$ for every χ' in relation R' to χ_0' . By definition of strong convergence, for every $\epsilon > 0$ there exists an χ_e'' such that

$\|\mu_{\chi''} - \mu_{\chi'}\| \leq \epsilon$ for every $\chi'' R \chi'$. From weak convergence we see that for every $\epsilon > 0$ there exists an χ'_ϵ such that $|(\mu_{\chi'_\epsilon} - \mu_1, \mu_2)| \leq \epsilon$ for every $\chi' R \chi'_\epsilon$. Let $\chi_{1\epsilon}$ be in relation R to both χ_0 and χ'_ϵ . Then $L_{\chi_1, \chi_{1\epsilon}}(\mu_{\chi_1}, \mu_{\chi_{1\epsilon}}) = (\mu_1, \mu_2)$. The other equalities follow readily if we note that for any convergent system of numbers a_χ , $|L_{\chi, \chi} a_\chi - a| \leq r$ when $|a_\chi - a| \leq r$ with $\chi R \chi_0$.

(4) follows immediately from (3).

3. The Fourier situation in a unitary space

In the immediate sequel we shall be concerned with an arbitrary subspace $\mathcal{M}_0$ of $\mathcal{M}$ ($\mathcal{M}$ is merely unitary in Theorems 4-8). Associated with $\mathcal{M}_0$ is an ϵ on $\mathcal{M}_0 \mathcal{M}_0$ to $\mathcal{A}$ such that $\epsilon(\mu_{01}, \mu_{02}) \equiv (\mu_{01}, \mu_{02})$ for μ_{01}, μ_{02} in $\mathcal{M}_0$. This matrix, being a gramian, is positive hermitian. And, according to the $\mathcal{A}P\epsilon$ -theory in general analysis it gives rise to modular functions μ' on $\mathcal{M}_0$ to $\mathcal{A}$ forming a space $\mathcal{M}'$ and an integration process J on $\mathcal{M}' \overline{\mathcal{M}'}$ to $\mathcal{A}$. For reasons that will appear in the sequel the space $\mathcal{M}'$ will be sometimes referred to as the Fourier corresponding space to $\mathcal{M}_0$.

Let σ be a finite set of points in $\mathcal{M}_0$, α on σ to $\mathcal{A}$, ξ' on $\mathcal{M}_0$ to $\mathcal{A}$. Then we define

$$K \alpha \sigma \equiv \sum_{\mu_1}^{\sigma} \sum_{\mu_2}^{\sigma} \alpha(\mu_1) \gamma_{\sigma}(\mu_1, \mu_2) \mu_2, \quad K \xi' \mathcal{M}_0 \equiv \int_{\sigma} K \xi' \sigma$$

γ_{σ} being the general reciprocal of $\epsilon(\sigma \sigma)$. We write $(\mu, \mathcal{M}_0)$ for (μ, μ_0) as μ_0 varies over $\mathcal{M}_0$ so that it is a function on $\mathcal{M}_0$ to $\mathcal{A}$.

THEOREM 4. (Best approximation). A best approximation to μ can be obtained from

$$\|\mu - K\alpha\sigma\|^2 = \|\mu\|^2 - J_\sigma(\mu, \sigma)(\sigma, \mu) + J_\sigma(\alpha - (\mu, \sigma))(\bar{\alpha} - (\sigma, \mu))$$

In particular, $\|K\alpha\sigma\|^2 = J_\sigma \alpha \bar{\alpha}$.

First we have equalities

$$\begin{aligned} \|\mu - K\alpha\sigma\|^2 &= (\mu - K\alpha\sigma, \mu - K\alpha\sigma) \\ &= \|\mu\|^2 - (K\alpha\sigma, \mu) - (\mu, K\alpha\sigma) + (K\alpha\sigma, K\alpha\sigma) \\ &= \|\mu\|^2 - J_\sigma \alpha(\sigma, \mu) - J_\sigma(\mu, \sigma) \bar{\alpha} + (K\alpha\sigma, K\alpha\sigma), \end{aligned}$$

Now

$$(K\alpha\sigma, K\alpha\sigma) = J_\sigma \alpha(\sigma, K\alpha\sigma) = J_\sigma \alpha(J_\sigma(\sigma, \sigma) \bar{\alpha}) = J_\sigma \alpha(J_\sigma \bar{\alpha}) = J_\sigma \alpha \bar{\alpha}.$$

Hence

$$\|\mu - K\alpha\sigma\|^2 = \|\mu\|^2 - J_\sigma \alpha(\sigma, \mu) - J_\sigma(\mu, \sigma) \bar{\alpha} + J_\sigma \alpha \bar{\alpha}.$$

Adding and subtracting $J_\sigma(\mu, \sigma)(\sigma, \mu)$ to the right member,

$$\|\mu - K\alpha\sigma\|^2 = \|\mu\|^2 + J_\sigma(\alpha - (\mu, \sigma))(\bar{\alpha} - (\sigma, \mu)) - J_\sigma(\mu, \sigma)(\sigma, \mu),$$

since

$$J_\sigma(\alpha - (\mu, \sigma))(\bar{\alpha} - (\sigma, \mu)) = J_\sigma(\mu, \sigma)(\sigma, \mu) - J_\sigma \alpha(\sigma, \mu) - J_\sigma(\mu, \sigma) \bar{\alpha} + J_\sigma \alpha \bar{\alpha}.$$

THEOREM 5. (Bessel inequality). The function (μ, π_0) is modular relative to ϵ for every μ in π and $M(\mu, \pi_0) \leq \|\mu\|$.

Consider an arbitrary function β on σ to $\mathcal{A}$ such that $\sum \sum \beta \in \bar{\beta} \leq 1$. We know that $|\sum(\mu, \sigma) \bar{\beta}| = |(\mu, \sum \beta \sigma)| \leq \|\mu\| \cdot \|\sum \beta \sigma\|$. Now $\|\sum \beta \sigma\|^2 = (\sum \beta \sigma, \sum \beta \sigma) = \sum \sum \beta(\sigma, \sigma) \bar{\beta} = \sum \sum \beta \in \bar{\beta} \leq 1$. Hence $|\sum(\mu, \sigma) \bar{\beta}| \leq \|\mu\|$ for every pair σ, β such that $\sum \sum \beta \in \bar{\beta} \leq 1$. So, (μ, π_0) is modular relative to ϵ and $M(\mu, \pi_0) \leq \|\mu\|$.

Hereafter we use $K_W \frac{1}{\epsilon} \pi_0$ to denote the limit $\lim_{\sigma} \frac{1}{\sigma} K \frac{1}{\epsilon} \sigma$.

THEOREM 6. (Equivalence of Fourier and Parseval equalities). Let μ^* be a given point in $\mathcal{M}$. Then the next four conditions on μ^* are mutually equivalent:

1. $\mu^* = K_W(\mu^*, \pi_0)\pi_0$
2. $(\mu, \mu^*) = J(\mu, \pi_0)(\pi_0, \mu^*)$ for every μ in $\mathcal{M}$
3. $(\mu^*, \mu^*) = J(\mu^*, \pi_0)(\pi_0, \mu^*)$
4. $\mu^* = K(\mu^*, \pi_0)\pi_0$

Here we use the scheme: $(1) \rightarrow (2) \rightarrow (3) \rightarrow (4) \rightarrow (1)$.

$$\begin{aligned}
 (1) \rightarrow (2): \quad (\mu, \mu^*) &= (\mu, \int_{\sigma} K(\mu^*, \sigma) \sigma) \\
 &= \int_{\sigma} (\mu, K(\mu^*, \sigma) \sigma) \\
 &= \int_{\sigma} J_{\sigma}(\mu, \sigma)(\sigma, \mu^*) \text{ by Theorem 1(3)} \\
 &= J(\mu, \pi_0)(\pi_0, \mu^*),
 \end{aligned}$$

since $(\mu, \pi_0), (\mu^*, \pi_0)$ are modular by the Bessel inequality.

$(2) \rightarrow (3)$: obvious.

$(3) \rightarrow (4)$ is readily seen from the best approximation formula, with $\alpha = (\mu^*, \sigma)$.

$(4) \rightarrow (1)$: obvious.

THEOREM 7. Let ξ' be a function on $\mathcal{M}_0$ to $\mathcal{A}$.

1. If $K_W \xi' \pi_0$ exists then

$$\int_{\sigma} J_{\sigma} \xi' \overline{\xi'} = \int_{\sigma} \|K \xi' \sigma\|^2 < \infty, \quad (K_W \xi' \pi_0, \mu) = J \xi'(\pi_0, \mu).$$

2. If $K \xi' \pi_0$ exists and is equal to μ^* , then

$$\int_{\sigma} J_{\sigma} \xi' \overline{\xi'} = \|\mu^*\|^2.$$

3. Suppose that ξ' is accordant relative to ϵ and $K_W \xi' \pi_0 = \mu^*$, then (a) ξ' is modular with respect to ϵ , (b) $\xi' = (\mu^*, \pi_0)$, and (c) $K \xi' \pi_0$ exists and equals μ^* .

In Theorem 4 put $\alpha = \xi'$, then $\|K \xi' \sigma\|^2 = J_\sigma \xi' \overline{\xi'}$ for every σ .

(1) As $K \xi' \sigma$ converges weakly, $\limsup \|K \xi' \sigma\|^2 < \infty$. Hence

$$\limsup J_\sigma \xi' \overline{\xi'} = \limsup \|K \xi' \sigma\|^2 < \infty.$$

From the weak convergence we have also

$$(\limsup K \xi' \sigma, \mu) = \limsup (K \xi' \sigma, \mu) = \limsup J_\sigma \xi' (\sigma, \mu).$$

(2) In case of strong convergence,

$$\|\mu\|^2 = \limsup \|K \xi' \sigma\|^2 = \limsup J_\sigma \xi' \overline{\xi'}.$$

(3) $(\mu^*, \pi_0) = (K_W \xi' \pi_0, \pi_0) = J \xi' (\pi_0, \pi_0) = J \xi' \epsilon = \xi'$, since ξ' is accordant with respect to ϵ , K_W a weak limit. By the Bessel inequality (μ^*, π_0) is modular. Hence ξ' is modular. As $K_W \xi' \pi_0 = \mu^*$ and $\xi' = (\mu^*, \pi_0)$, $K_W(\mu^*, \pi_0) \pi_0 = \mu^*$, which is a weak Fourier equality. Hence an equivalence in Theorem 6 shows the strong convergence.

THEOREM 8. Let $\pi_0 = \pi$ then we have the following simple results:

1. (μ, π) is modular relative to ϵ .
2. $K(\mu, \sigma) \sigma = \mu$ whenever σ contains μ .
3. If $K_W \mu' \pi = \mu$ then $(\mu, \pi) = \mu'$ where μ' as before is a point in π' .
4. $J(\mu_1, \pi)(\pi, \mu_2) = (\mu_1, \mu_2)$.
5. $\|\mu\| = M(\mu, \pi)$.

(1) is a special case of Theorem 5.

(2) By best approximation formula,

$$\|K(\mu, \sigma) \sigma - \mu\|^2 = \|\mu\|^2 - J_\sigma(\mu, \sigma)(\sigma, \mu) = \epsilon(\mu, \mu) - J_\sigma \epsilon(\mu, \sigma) \epsilon(\sigma, \mu)$$

$$= \epsilon(\mu, \mu) - \epsilon(\mu, \mu) \text{ when } \mu \text{ belongs to } \sigma.$$

Thus, $K(\mu, \sigma)\sigma = \mu$ for μ in σ .

(3) is already proved in (3) (b) of Theorem 7.

(4) From (2) we write $K(\mu_2, \pi)\pi = \mu_2$. Hence by equivalence of the Fourier and Parseval equalities, $(\mu_1, \mu_2) = J(\mu_1, \pi)(\pi, \mu_2)$.

(5) is a special case of (4).

4. General Cauchy condition for convergence; convergence of Fourier systems

For Section 4 the space π is supposed to be complete.

THEOREM 9. Let $\{\mu_\ell\}$ be an $\mathcal{LR}$ -system of points in π such that

$$\chi_1 \chi_2^L \|\mu_{\chi_1} - \mu_{\chi_2}\| = 0.$$

Then there exists one and only one point μ_* in π such that

$$\lim_{\chi} \mu_{\chi} = \mu_*.$$

To every μ of π corresponds a μ' in π' according to Theorem 8 such that $\mu' = (\mu, \pi)$. Consider $\mu'_{\chi} = (\mu_{\chi}, \pi)$. By (5) of Theorem 8,

$$\chi_1 \chi_2^L M(\mu'_{\chi_1} - \mu'_{\chi_2}) = 0.$$

By the $\mathcal{LPE}$ -theory there exists uniquely a numerical limit $\xi' = L_{\chi} \mu'_{\chi}$ which is modular relative to ϵ and such that

$$\lim_{\chi} M(\mu'_{\chi} - \xi') = 0.$$

Hence for every integer $m \geq 1$ there exists an χ_m such that

$M(\mu'_{\chi_m} - \xi') \leq 1/m$. So there is a system χ_m ($m = 1, 2, 3, \dots$)

such that $L_m M(\mu'_\chi - \xi') = 0$, which gives $L_{mn} M(\mu'_\chi - \mu'_\chi) = 0$.

Now, since $\mu'_\chi = (\mu_\chi, \mathfrak{M})$, we have $K\mu'_\chi \mathfrak{M} = \mu_\chi$ by (2) of Theorem 8. Further $M(\mu'_\chi - \mu'_\chi) = \|\mu_\chi - \mu_\chi\|$ by (5) of Theorem 8. Consequently $L_{mn}\|\mu_\chi - \mu_\chi\| = 0$. This, on account of the completeness of $\mathfrak{M}$, shows that there exists uniquely a point μ^* of $\mathfrak{M}$ such that $L_m \|\mu_\chi - \mu^*\| = 0$.

Let $\mu'^* = (\mu^*, \mathfrak{M})$. Then $L_m M(\mu'_\chi - \mu'^*) = 0$ in virtue of Theorem 8, (5). Hence $\xi' = \mu'^*$ by unicity of a strong limit. As $L_\chi M(\mu'_\chi - \xi') = 0$ we have $L_\chi M(\mu'_\chi - \mu'^*) = 0$. Again by Theorem 8, (5) we conclude that $L_\chi \|\mu_\chi - \mu^*\| = 0$.

THEOREM 10. Let $\mathfrak{M}_0$ be an arbitrary subspace of $\mathfrak{M}$ and $\mathfrak{M}'$ the Fourier corresponding space of $\mathfrak{M}_0$. Then, for every μ' of $\mathfrak{M}'$, $K\mu'\mathfrak{M}_0$ defined to be $\frac{1}{\sigma} K\mu'\sigma$ exists as a point of $\mathfrak{M}$.

By direct calculation,

$$\|K\mu'\sigma_2 - K\mu'\sigma_1\|^2 = \|K\mu'\sigma_2\|^2 + \|K\mu'\sigma_1\|^2 - [(K\mu'\sigma_2, K\mu'\sigma_1) + (K\mu'\sigma_1, K\mu'\sigma_2)]$$

Now from Theorem 4, $\|K\mu'\sigma_1\|^2 = J_{\sigma_1} \mu' \overline{\mu'}$, $\|K\mu'\sigma_2\|^2 = J_{\sigma_2} \mu' \overline{\mu'}$.

Also

$$\begin{aligned} (K\mu'(\sigma+\tau), K\mu'\sigma) &= J_{\sigma+\tau} J_\sigma \mu'(\sigma+\tau, \sigma) \overline{\mu'} = J_{\sigma+\tau} J_\sigma \mu' \in \overline{\mu'} \\ &= J_\sigma \mu'_{\sigma+\tau} \overline{\mu'} = J_\sigma \mu'_\sigma \overline{\mu'_\sigma} = J_\sigma \mu' \overline{\mu'}, \end{aligned}$$

where τ is an arbitrary finite set of points (in $\mathfrak{M}_0$) distinct from those in σ . By similar computation we find $(K\mu'\sigma, K\mu'(\sigma+\tau)) = J_\sigma \mu' \overline{\mu'}$. These lead to

$$\|K\mu'(\sigma+\tau) - K\mu'\sigma\|^2 = J_{\sigma+\tau} \mu' \overline{\mu'} + J_\sigma \mu' \overline{\mu'} - (J_\sigma \mu' \overline{\mu'} + J_\sigma \mu' \overline{\mu'}) = J_{\sigma+\tau} \mu' \overline{\mu'} - J_\sigma \mu' \overline{\mu'}.$$

From the triangle inequality we see that

$$\begin{aligned} \|K\mu'\sigma_2 - K\mu'\sigma_1\| &\leq \|K\mu'\sigma_2 - K\mu'(\sigma_1 + \sigma_2)\| + \|K\mu'(\sigma_1 + \sigma_2) - K\mu'\sigma_1\| \\ &\leq (J_{\sigma_1 + \sigma_2} \mu' \overline{\mu'} - J_{\sigma_2} \mu' \overline{\mu'})^{1/2} + (J_{\sigma_1 + \sigma_2} \mu' \overline{\mu'} - J_{\sigma_1} \mu' \overline{\mu'})^{1/2}. \end{aligned}$$

But $L_{\sigma} J_{\sigma} \mu' \overline{\mu'} = J \mu' \overline{\mu'}$ so that $L_{\sigma} J_{\sigma} |J_{\sigma} \mu' \overline{\mu'} - J_{\tau} \mu' \overline{\mu'}| = 0$. We have hence $L_{\sigma} J_{\sigma} |J_{\sigma+\tau} \mu' \overline{\mu'} - J_{\tau} \mu' \overline{\mu'}| = 0$. Therefore $L_{\sigma_1 \sigma_2} \|K \mu' \sigma_2 - K \mu' \sigma_1\| = 0$. This, by Theorem 9, shows that the system $K \mu' \sigma$ converges strongly to a limit in $\mathcal{M}$ which we denote by $K \mu' \pi_0$.

5. Completeness and compactness of a space

Definitions. (1) A unitary space $\mathcal{M}$ is said to be hypercomplete if every strongly convergent system $\{\mu_{\chi}\}$ has a strong limit in $\mathcal{M}$. It is called weakly hypercomplete in case weakly convergent systems have weak limits in $\mathcal{M}$.

2. A complete space is one in which every strongly convergent sequence $\{\mu_m\}$ has a strong limit. If for every weakly convergent sequence $\{\mu_m\}$ there exists a weak limit then $\mathcal{M}$ is weakly complete.

3. We call a space compact in case for every infinite sequence $\{\mu_m\}$ with $\exists \|\mu_m\| < \infty$ there exists an infinite subsequence $\{\mu_{m_n}\}$ converging weakly to a μ^* in $\mathcal{M}$.

THEOREM 11. Let $\mathcal{M}$ be a unitary space. Then the next five conditions on $\mathcal{M}$ are mutually equivalent: (1) $\mathcal{M}$ is complete, (2) $\mathcal{M}$ is hypercomplete, (3) $\mathcal{M}$ is weakly hypercomplete, (4) $\mathcal{M}$ is weakly complete, (5) $\mathcal{M}$ is compact.

(1) $\rightarrow$ (2) follows from Theorem 9.

(2) $\rightarrow$ (3). Let $\mu' \equiv (\mu, \mathcal{M})$. By hypothesis, $L_{\chi} \|\mu_{\chi}\| < \infty$, $L_{\chi_1 \chi_2} (\mu_{\chi_1} - \mu_{\chi_2}, \mu) = 0$ for every μ . From Theorem 8, $J \mu'_{\chi_1} \overline{\mu'_{\chi_2}} = (\mu_{\chi_1}, \mu_{\chi_2})$. Hence $\{\mu'_{\chi}\}$ converges weakly to a modular μ'_0 by a theorem of the *alpe*-theory. According to Theorem 10, $K \mu' \pi$ exists because $\mathcal{M}$ is hypercomplete by (2). Let $\mu^* \equiv K \mu'_0 \pi$. Then $\mu'_0 = (\mu^*, \mathcal{M})$ by Theorem 8 (3). Hence by

Theorem 8 (4) $(\mu^*, \mu) = J \mu'_0 \overline{\mu'}$. We have consequently
 $(\mu_\chi - \mu^*, \mu) = J(\mu'_\chi - \mu'_0) \overline{\mu'}$. Thus $L_\chi(\mu_\chi - \mu^*, \mu) = 0$ for every μ . So π is weakly hypercomplete.

(3) $\rightarrow$ (4) is obvious.

(4) $\rightarrow$ (1). Consider a sequence $\{\mu_m\}$ such that $L_{mn} \|\mu_m - \mu_n\| = 0$. As before, let $\mu' \equiv (\mu, \pi)$. Then by Theorem 8, (4), $L_{mn} M(\mu'_m - \mu'_n) = 0$. From a theorem of the $\alpha\mathcal{P}\mathcal{E}$ -theory this shows that there exists a μ'_* such that $L_m M(\mu'_m - \mu'_*) = 0$. Since by (4) π is weakly complete, $L_m \mu_m$ exists and let it be μ_0 . Now $\mu'_0 \equiv (\mu_0, \pi) = (L_m \mu_m, \pi) = L_m(\mu_m, \pi) = L_m \mu'_m$ by definition of μ'_m . As $L_m \mu'_m = \mu'_*$, $\mu'_0 \equiv (\mu_0, \pi) = \mu'_*$. Hence $J \mu'_* \overline{\mu'_*} = (\mu_0, \mu_0)$ by Theorem 8, (4). Therefore $M(\mu'_m - \mu'_*) = M(\mu'_m - \mu'_0) = \|\mu_m - \mu_0\|$, and $L_m \|\mu_m - \mu_0\| = 0$.

(2) $\rightarrow$ (5). As $M \mu'_m = \|\mu_m\|$ are uniformly bounded, there exists by a theorem of the $\alpha\mathcal{P}\mathcal{E}$ -theory an infinite subsequence $\{\mu'_{m_n}\}$ and a μ'_0 such that $L_n J \mu'_{m_n} \overline{\mu'} = J \mu'_0 \overline{\mu'}$ for every μ' in π' . But $(\mu_m, \mu) = J \mu'_m(\pi, \mu)$ by Theorem 8, (4). Hence $L_n(\mu_{m_n}, \mu) = J \mu'_0(\pi, \mu)$. Now, since π is hypercomplete by (2), $K \mu'_0 \pi$ exists. Let $K \mu'_0 \pi \equiv \mu^*$. Then $(\mu^*, \pi) = \mu'_0$ by Theorem 8, (5). By virtue of Theorem 8 (4) it is obvious that $(\mu^*, \mu) = J(\mu^*, \pi)(\pi, \mu) = J \mu'_0(\pi, \mu)$. Consequently we have $L_n(\mu_{m_n}, \mu) = (\mu^*, \mu)$ for every μ . Thus π is compact.

(5) $\rightarrow$ (4). As $L_m \|\mu_m\|$ is finite, $\|\mu_m\|$ are uniformly bounded. By compactness of π in (5) there exists an infinite subsequence $\{\mu_{m_n}\}$ and a μ_0 such that $L_n(\mu_{m_n}, \mu) = (\mu_0, \mu)$ for every μ . Now $L_n(\mu_{m_n}, \mu) = L_m(\mu_m, \mu)$ since $L_m(\mu_m, \mu)$ exists from the hypothesis that $L_{mn}(\mu_m - \mu_n, \mu) = 0$. Hence $(\mu_0, \mu) = L_m(\mu_m, \mu)$. So, π is weakly complete.

6. Convergent systems in a complete space

The following two theorems are corollaries of Theorem 11.

THEOREM 12. If $L_\chi \|\mu_\chi\|$ is finite and $L_{\chi_1 \chi_2} (\mu_{\chi_1} - \mu_{\chi_2}, \mu) = 0$ for every μ , then there exists uniquely a μ_0 such that $L_\chi \mu_\chi = \mu_0$.

THEOREM 13. If $\{\mu_m\}$ is an infinite sequence of which $\|\mu_m\|$ are uniformly bounded, then there exists an infinite subsequence $\{\mu_{m_n}\}$ and a μ_0 such that $L_{\chi_n} \mu_{m_n} = \mu_0$.

7. The Fourier situation in a complete unitary space

Consider an arbitrary subspace $\mathcal{M}_0$ of $\mathcal{M}$. To simplify writing let us denote $K(\mu, \mathcal{M}_0)\mathcal{M}_0$ by $P_0\mu$ so that P_0 is a function on $\mathcal{M}$ to $\mathcal{M}$.

THEOREM 14. For every μ we have $P_0(P_0\mu) = P_0\mu$.

Let μ^* be $P_0\mu$. Then $P_0\mu^* = K(\mu^*, \mathcal{M}_0)\mathcal{M}_0$. Now $(\mu^*, \mathcal{M}_0) = (K(\mu, \mathcal{M}_0)\mathcal{M}_0, \mathcal{M}_0) = J(\mu, \mathcal{M}_0)(\mathcal{M}_0, \mathcal{M}_0) = J(\mu, \mathcal{M}_0) \in (\mu, \mathcal{M}_0)$. Thus $P_0\mu^* = K(\mu, \mathcal{M}_0)\mathcal{M}_0 = P_0\mu$ and $P_0(P_0\mu) = P_0\mu^*$.

Definition. A complete linear unitary subspace $\mathcal{M}^*$ of $\mathcal{M}$ is called the complete linear extension of $\mathcal{M}_0$ in case (1) $\mathcal{M}^*$ contains $\mathcal{M}_0$ and (2) any complete linear subspace containing $\mathcal{M}_0$ contains $\mathcal{M}^*$. In case $\mathcal{M}^*$ is equal to $\mathcal{M}$, $\mathcal{M}_0$ is said to be dense on $\mathcal{M}$.

It is evident that the greatest common subspace of all complete subspaces of $\mathcal{M}$ containing $\mathcal{M}_0$ is a complete linear space and is obviously $\mathcal{M}^*$. If we make similar definitions for compact extensions and other types of complete extension, these extensions always coincide because of Theorem 11.

THEOREM 15. An element μ belongs to $\mathcal{M}^*$ if and only if it is of form $\mu = \sum_n \mu_{0L_n}$ where the μ_{0L} 's are finite linear

combinations of elements in $\mathcal{M}_0$. Moreover, $\mathcal{M}^* = P_0\mathcal{M}$.

Define $\mathcal{M}_1$ to be the set of all μ_1 's of the form $\mu_1 = \sum_n \mu_{0Ln}$. Then the theorem may be written symbolically thus: $\mathcal{M}^* = \mathcal{M}_1 = P_0\mathcal{M}$.

$\mathcal{M}^* \supset \mathcal{M}_1$: Since $\mathcal{M}^*$ is linear complete and μ_{0Ln} are in $\mathcal{M}^*$, μ_1 belongs to $\mathcal{M}^*$. Hence $\mathcal{M}^* \supset \mathcal{M}_1$.

$\mathcal{M}^* \subset \mathcal{M}_1$: We wish to show that $\mathcal{M}_1$ is linear complete. Consider a sequence $\{\mu_{1n}\}$ in $\mathcal{M}_1$ and a μ^* such that $\sum_n \mu_{1n} = \mu^*$. Since μ_{1n} are in $\mathcal{M}_1$, there exists a double sequence $\{\mu_{0Lnm}\}$ such that $\mu_{1n} = \sum_m \mu_{0Lnm}$ for every n . Take the sequence of numbers $1/k$, $k = 1, 2, 3, \dots$. Then for every k there exists an n_k such that $\|\mu^* - \mu_{1n}\| < 1/2k$ for every $n \geq n_k$. Further, for every k there exists an m_k such that $\|\mu_{1n_k} - \mu_{0Ln_k m}\| < 1/2k$ for every $m \geq m_k$. So, for every k there exists a pair (n_k, m_k) such that $\|\mu^* - \mu_{0Ln_k m_k}\| < 1/k$. Consequently $L_k \|\mu^* - \mu_{0Ln_k m_k}\| = 0$, i.e., $\mu^* = \sum_k \mu_{0Ln_k m_k}$. Thus μ^* belongs to $\mathcal{M}_1$ and $\mathcal{M}_1$ is linear complete.

$P_0\mathcal{M} \subset \mathcal{M}_1$: By definition $P_0\mu = K(\mu, \mathcal{M}_0)\mathcal{M}_0$ so that $P_0\mu$ is of form $\sum_\sigma \mu_{0L\sigma}$. But we can easily choose a system $\{\mu_{0Ln}\}$ out of $\{\mu_{0L\sigma}\}$ such that $P_0\mu = \sum_n \mu_{0Ln}$.

$P_0\mathcal{M} \supset \mathcal{M}^*$: We shall show that $P_0\mathcal{M}$ is complete. Suppose ξ_m are in $P_0\mathcal{M}$. Then there exist μ_m such that $P_0\mu_m = \xi_m$. Since $L_{mn} \|\xi_m - \xi_n\| = 0$ and $\mathcal{M}$ is complete, there exists a ξ^* such that $\sum_m \xi_m = \xi^*$. Hence $\sum_m P_0\xi_m = P_0\xi^*$ as $\|P_0\mu\| \leq \|\mu\|$ by the Bessel inequality. But $P_0(P_0\mu) = P_0\mu$ from Theorem 14. It follows that $P_0\xi^* = \sum_m P_0\xi_m = \sum_m P_0(P_0\mu_m) = \sum_m P_0\mu_m = \sum_m \xi_m = \xi^*$. Now $P_0\xi^*$ is in $P_0\mathcal{M}$. So $\xi^* = P_0\xi^*$ belongs to $P_0\mathcal{M}$. Thus $P_0\mathcal{M}$ is complete.

THEOREM 16. (Fourier equality). For every μ^* in $\mathcal{M}^*$ it is true that

$$K(\mu^*, \mathcal{M}_0)\mathcal{M}_0 = \mu^* = P_0\mu^*.$$

By Theorem 15 there exist a μ such that $P_0\mu = \mu^*$.

Hence $K(P_0\mu, \mathcal{M}_0)\mathcal{M}_0 = P_0(P_0\mu) = P_0\mu = \mu^*$ from Theorem 14.

THEOREM 17. (Parseval equality). Let μ^* be a point in $\mathcal{M}^*$. Then for every μ we have

$$(\mu, \mu^*) = J(\mu, \mathcal{M}_0)(\mathcal{M}_0, \mu^*).$$

This is an immediate consequence from Theorem 16 on account of the equivalence between the Fourier, Parseval equalities

THEOREM 18. The following conditions are equivalent:

(1) $\mathcal{M}^* = \mathcal{M}$, (2) $\mu = P_0\mu$ for every μ in $\mathcal{M}$, (3) $(\mu_1, \mu_2) = J(\mu_1, \mathcal{M}_0)(\mathcal{M}_0, \mu_2)$ for every pair μ_1, μ_2 in $\mathcal{M}$, (4) if $(\mu, \mu_0) = 0$ for every μ_0 in $\mathcal{M}_0$, then $\mu = 0$, (5) if $(\mu_1, \mathcal{M}_0) = (\mu_2, \mathcal{M}_0)$ then $\mu_1 = \mu_2$, and (6) every element μ in $\mathcal{M}$ is of form $K\mu'\mathcal{M}_0$ where μ' belongs to $\mathcal{M}'$.

(1) $\rightarrow$ (2). From the Fourier equality $\mu^* = P_0\mu^*$. But by (1), $\mathcal{M}^* = \mathcal{M}$. Hence $\mu = P_0\mu$ for every μ in $\mathcal{M}$.

(2) $\rightarrow$ (3). By (2) we have the Fourier equality for every μ in $\mathcal{M}$. So, according to the equivalence of the Fourier, Parseval equalities we have the equality given in (3) for every pair μ_1, μ_2 in $\mathcal{M}$.

(3) $\rightarrow$ (4). It is given that $(\mu, \mu_0) = 0$ for every μ_0 . Then by (3), $(\mu, \mu) = J(\mu, \mathcal{M}_0)(\mathcal{M}_0, \mu) = 0$. Thus $\mu = 0$.

(4) $\rightarrow$ (5). As $(\mu_1, \mathcal{M}_0) = (\mu_2, \mathcal{M}_0)$, $(\mu_1 - \mu_2, \mu_0) = 0$ for every μ_0 in $\mathcal{M}_0$. By (4), $\mu_1 - \mu_2 = 0$.

(5) $\rightarrow$ (6). Define $\mu' = (\mu, \pi_0)$, $\mu \equiv K(\mu, \pi_0)\pi_0$. Then $(\mu, \pi_0) = (K(\mu, \pi_0)\pi_0, \pi_0) = J(\mu, \pi_0)E = (\mu, \pi_0)$. Hence by (5), $\mu = \mu$. Thus for every μ there exists a μ' such that $\mu = K\mu'\pi_0$.

(6) $\rightarrow$ (1). By Theorem 15, $\pi_* = P_0\pi = \pi_1$. If $\mu = K\mu'\pi_0$, μ is of form $\mu = \sum_n \mu_{0Ln}$, so that μ belongs to π_* . So, $\pi_* = \pi$.

THEOREM 19. An element ξ belongs to π_* if and only if it satisfies one of the ensuing equivalent conditions:

(1) $\xi = K\mu'\pi_0$, (2) $\xi = P_0\mu$, (3) $\xi = P_0\xi$, (4) $(\xi, \xi) = (\xi, P_0\xi)$, (5) $\|\xi\| = M(\xi, \pi_0)$.

Here the proof need not be given as it can be readily supplied by the reader.

THEOREM 20. (Riesz-Fischer theorem). For every modular function μ' in π' there exists one and only one μ_* in π_* such that $\mu' = (\mu_*, \pi_0)$.

Define $\mu_* \equiv K\mu'\pi_0$ which exists as a strong limit, since π is complete. Then by Theorem 7, (3), $\mu' = (\mu_*, \pi_0)$. To show unicity, suppose $\mu' = (\mu_{*1}, \pi_0) = (\mu_{*2}, \pi_0)$. This, according to Theorem 18, (5), implies $\mu_{*1} = \mu_{*2}$.

8. Orthogonal decomposition relative to a subspace

Definitions. A vector μ_1 is said to be orthogonal to another μ_2 in case $(\mu_1, \mu_2) = 0$ and we write $\mu_1 \perp \mu_2$. If every vector of π_1 is orthogonal to every one in π_2 then π_1 is orthogonal to π_2 . We use $0\pi_0$ to denote the set of all vectors orthogonal to π_0 .

THEOREM 21. (1) For every μ in π there exists uniquely a pair of points μ_1 in π_* and μ_2 in $0\pi_*$ such that $\mu = \mu_1 + \mu_2$.

Further, $\mu_1 = K(\mu, \pi_0)\pi_0$.

(2) For every μ' the general solution of $\mu' = (\mu, \pi_0)$ for μ is $\mu = K\mu'\pi_0 + 0\pi_*$.

(1) Let $\mu_1 \equiv P_0\mu$ so that μ_1 is in π_* . Further $\mu_2 \equiv \mu - \mu_1$ is orthogonal to π_* , since $(\mu_2, \mu_*) = (\mu, \mu_*) - (P_0\mu, \mu_*) = (\mu, \mu_*) - (P_0\mu, P_0\mu_*) = (\mu, \mu_*) - (\mu, \mu_*) = 0$ by Theorem 19, (4) and the Parseval equality. Suppose $\mu = \mu_1 + \mu_2$ with μ_1 in π_* and μ_2 in $0\pi_*$. Then, since $\mu_1 + \mu_2 = \mu_1 + \mu_2$, $\mu_1 - \mu_1 = \mu_2 - \mu_2$ is in π_* and $0\pi_*$. This can be so only when $\mu_1 - \mu_1 = \mu_2 - \mu_2 = 0$.

(2) Suppose μ is a solution for μ' in $\mu' = (\mu, \pi_0)$. Then $\mu' = (\mu, \pi_0)$. Hence $\mu - K\mu'\pi_0 = \mu - K(\mu, \pi_0)\pi_0$ is in $0\pi_*$ by (1). Thus a solution is of form $\mu = K\mu'\pi_0 + 0\pi_*$. On the other hand, $(K\mu'\pi_0, \pi_0) + (0\pi_*, \pi_0) = (K\mu'\pi_0, \pi_0) = J\mu'(\pi_0, \pi_0) = J\mu'\epsilon = \mu'$. So $K\mu'\pi_0 + 0\pi_*$ is a solution.

Let π be a complete unitary space. Consider all the non-zero vectors of π . Well order them and obtain the system $\{\mu_\alpha\}$. Define

$$\xi_0 \equiv \mu_0, \quad \xi_\alpha \equiv \mu_\alpha - \sum_{\beta < \alpha} \frac{1}{\|\xi_\beta\|} (\mu_\alpha, \xi_\beta) \xi_\beta, \quad \alpha \neq 0,$$

where $\frac{1}{\|\xi_\beta\|}$ is taken to be zero when $\|\xi_\beta\| = 0$, and the general sum converges strongly. Then it can be easily verified that ξ_0, ξ_1 are orthogonal. To make an induction proof let us suppose that $\{\xi_\beta\}$, $\beta < \alpha$, is an orthogonal set. By the decomposition theorem just proved, the vector

$$\mu_\alpha - P_{\{\xi_\beta\}} \mu_\alpha$$

is orthogonal to every ξ_β , $\beta < \alpha$. But the ξ 's are assumed to be orthogonal, so that

$$P_{\{\xi_\beta\}} \mu_\alpha = \sum_{\beta < \alpha} \frac{1}{\|\xi_\beta\|} (\mu_\alpha, \xi_\beta) \xi_\beta.$$

Consequently, ξ_α is orthogonal to every ξ_β , $\beta < \alpha$, and hence $\{\xi_\beta\}$, $\beta \leq \alpha$, is an orthogonal set. Then according to the induction principle $\{\xi_\beta\}$, $\beta \leq \alpha$, is an orthogonal set for every α . Thus, ξ_α is orthogonal to every other vector ξ_γ , $\alpha \neq \gamma$. Further, every μ different from zero is expressible as a general sum of some ξ 's and consequently they form a complete basis for $\mathcal{M}$, that is, there exists no non-zero vector ξ of $\mathcal{M}$ orthogonal to all the ξ 's. Thus $\{\xi_\alpha\}$ is dense on $\mathcal{M}$. Pick out the non-zero ξ 's and normalize them as usual by writing $\xi/\|\xi\|$. The new set will be dense on $\mathcal{M}$.

THEOREM 22. (Existence of orthogonal basis). Every complete unitary space, not entirely zero, has a complete orthonormal basis.

The theorem was given by Moore in general analysis for the space of modular functions. In discussing some results in the present paper von Neumann first raised the existence question of orthogonal basis. Later he established this theorem, using set-theoretic arguments. While these proofs and the one just given differ in their motivations, they all make use of the Zermelo axiom and the consequent normal order theorem.

9. Representation of a linear functional

Definitions. A function L on $\mathcal{M}$ to $\mathcal{A}$ is linear when $L(\mu_1 + \mu_2) = L(\mu_1) + L(\mu_2)$ and $L(a\mu) = aL(\mu)$. We call a function F on $\mathcal{M}$ to $\mathcal{A}$ bounded in case there exists a constant number s such that $|F(\mu)| \leq s\|\mu\|$ for every μ in $\mathcal{M}$. The greatest lower bound of such numbers is known as its modulus MF.

THEOREM 23. Let L be a bounded linear functional on $\mathcal{M}$ and $\mathcal{M}_0$ be a subspace dense in $\mathcal{M}$. Then there exists uniquely a μ^* such that $L(\mu) = (\mu, \mu^*)$ for every μ . Further $\mu^* = \overline{KL(\mathcal{M}_0)}\mathcal{M}_0$.

We first show that $\overline{L(\mathcal{M}_0)}$ is modular with respect to $\epsilon \equiv (\mathcal{M}_0, \mathcal{M}_0)$. To do this let σ be a finite set of points in $\mathcal{M}_0$. Suppose β is on σ to α with $\sum_{\sigma} \sum_{\beta \in \bar{\sigma}} \beta \in \bar{\sigma} \leq 1$. We know

$$\left| \sum_{\mu_0} \overline{L(\mu_0)} \beta(\mu_0) \right| = \left| \sum_{\mu_0} \beta(\mu_0) L(\mu_0) \right| = \left| \overline{L(\sum_{\sigma} \beta \sigma)} \right| = \left| L(\sum_{\sigma} \beta \sigma) \right| \leq ML \cdot \left\| \sum_{\sigma} \beta \sigma \right\|.$$

As $\left\| \sum_{\sigma} \beta \sigma \right\|^2 = \sum_{\sigma} \sum_{\beta \in \bar{\sigma}} \beta \in \bar{\sigma} \leq 1$, $\left\| \sum_{\sigma} \beta \sigma \right\| \leq 1$. Hence

$$\left| \sum_{\mu_0} \overline{L(\mu_0)} \beta(\mu_0) \right| \leq ML \text{ for every pair } \sigma, \beta \text{ such that } \sum_{\sigma} \sum_{\beta \in \bar{\sigma}} \beta \in \bar{\sigma} \leq 1.$$

Thus $\overline{L(\mathcal{M}_0)}$ is modular relative to ϵ . Since $\overline{L(\mathcal{M}_0)}$ is modular,

$\overline{KL(\mathcal{M}_0)}\mathcal{M}_0$ exists. Call it μ^* . Let μ_0 be a point in $\mathcal{M}_0$ then

$$(\mu_0, \mu^*) = (\mu_0, \overline{KL(\mathcal{M}_0)}\mathcal{M}_0) = J(\mu_0, \mathcal{M}_0) \overline{L(\mathcal{M}_0)} = L(\mu_0). \text{ And}$$

on account of linearity of L , $(\xi, \mu^*) = L(\xi)$, where ξ is a

finite linear combination of elements in $\mathcal{M}_0$. Consider a $\mu =$

$$\sum_n \xi_n. \text{ Then } (\mu, \mu^*) = (\sum_n \xi_n, \mu^*) = L_n(\xi_n, \mu^*) = L_n L(\xi_n).$$

Now $|L(\xi_n) - L(\mu)| \leq ML \cdot \left\| \xi_n - \mu \right\|$. Hence $L_n L(\xi_n) = L(\mu)$ and

so $(\mu, \mu^*) = L(\mu)$.

CHAPTER II

OPERATORS

1. Operators and their algebraic operations

Let us consider a linear space $\mathcal{M}$. An operator Ω in $\mathcal{M}$ is a single-valued function on a subspace $\mathcal{M}(\Omega)$ to $\mathcal{M}$. If $\mathcal{M}(\Omega)$ is linear and $\Omega(a\mu + b\zeta) = a\Omega\mu + b\Omega\zeta$ for every pair μ, ζ in $\mathcal{M}(\Omega)$, then Ω is said to be linear. Two operators occurring frequently are the identity I and the zero Z operators: $I\mu = \mu$, $Z\mu = 0$ for every μ .

We write $\Phi = \Omega + \Psi$ over the intersection $\mathcal{N}$ of $\mathcal{M}(\Omega)$ and $\mathcal{M}(\Psi)$ in case $\Phi\zeta = \Omega\zeta + \Psi\zeta$ for every ζ in $\mathcal{N}$. Similarly $\Phi = \Omega - \Psi$, $\Phi = a\Omega$ stand for $\Phi\zeta = \Omega\zeta - \Psi\zeta$ over $\mathcal{N}$ and $\Phi\zeta = a\Omega\zeta$ over $\mathcal{M}(\Omega)$. As usual we define the product $\Phi\Omega$ on $\mathcal{N}$ to be $\Phi(\Omega\zeta)$ over a subspace $\mathcal{N}$ of $\mathcal{M}(\Omega)$ with $\Omega\mathcal{N}$ lying in $\mathcal{M}(\Phi)$. Two operators Ω, Φ are permutable over $\mathcal{N}$ if $\Omega\mathcal{N}$ lies in $\mathcal{M}(\Phi)$ and $\Phi\mathcal{N}$ in $\mathcal{M}(\Omega)$ and if $\Omega\Phi = \Phi\Omega$ over $\mathcal{N}$. We call Γ a reciprocal of Ω if $\Gamma\Omega = I$ as on $\mathcal{M}(\Omega)$ and $\Omega\Gamma = I$ as on $\mathcal{M}(\Gamma)$.

2. Elementary metric properties of operators

For further considerations we need the idea of the modulus of an operator. It is the least positive (or zero) number m such that $\|\Omega\zeta\| \leq m\|\zeta\|$ for all ζ in $\mathcal{M}(\Omega)$ and we denote this by $M\Omega$. In case $M\Omega$ is finite we call Ω bounded. From this definition it follows at once that $M(\Phi\Omega) \leq M\Phi M\Omega$ where Φ, Ω are bounded operators. If $\mathcal{M}$ is a complete linear metric space, we

say that Ω is closed when for two strongly convergent sequences $\{\xi_n\}, \{\Omega\xi_n\}$ we have $L_n \xi_n$ in $\mathcal{M}(\Omega)$ and $\Omega L_n \xi_n = L_n \Omega \xi_n$.

Let Ω be defined over the entire complete unitary space $\mathcal{M}$. Then Ω is said to be continuous if $\Omega\mu_n$ converge strongly to $\Omega\mu$ whenever μ_n converge to μ in the strong sense. If a linear operator Ω is bounded then it is uniformly continuous. This can be seen readily and requires no proof. Conversely, if Ω is continuous it is bounded. To show this we use an indirect proof of Fréchet. Suppose that Ω is not bounded. Then, for every n there exists a μ_n with $\|\mu_n\| \leq 1$ such that $\|\Omega\mu_n\| > n$. Introduce $\xi_n = \mu_n/\sqrt{n}$ so that $L_n \|\xi_n\| = 0$. Due to the continuity of Ω , $\|\Omega\xi_n\|$ is uniformly bounded. But $\|\Omega\xi_n\| = \|\Omega\mu_n\|/\sqrt{n}$ so that $\|\Omega\xi_n\| > \sqrt{n}$. This contradiction leads to the desired converse. Thus we have

THEOREM 1. Boundedness, continuity, and uniform continuity are all equivalent conditions on a linear operator in a complete unitary space.

We have $|(\Omega\mu, \xi)| \leq \|\Omega\mu\| \leq M\Omega$ for $\|\mu\| \leq 1, \|\xi\| \leq 1$. But on the other hand $|(\Omega\mu, \xi)| \leq B|(\Omega\mu, \xi)|$ where μ, ξ again satisfy the unitary condition. Put $\xi = \Omega\mu/\|\Omega\mu\|$. Then $(\Omega\mu, \xi) = \|\Omega\mu\|$ and hence $\|\Omega\mu\| \leq B|(\Omega\mu, \xi)|$. This proves

THEOREM 2. For any linear operator in a unitary space the equality $M\Omega = B|(\Omega\mu, \xi)|, \|\mu\| \leq 1, \|\xi\| \leq 1$ holds.

For a given ξ the inner product $(\Omega\mu, \xi)$ (Ω being linear) defines a linear functional on $\mathcal{M}(\Omega)$ whose complete linear extension is assumed to be $\mathcal{M}$. If $|(\Omega\mu, \xi)|$ is bounded for $\|\mu\| \leq 1$, then, by the representation theorem of linear functionals in Chapter I there exists uniquely a ξ^* such that $(\Omega\mu, \xi) = (\mu, \xi^*)$ for every μ in $\mathcal{M}(\Omega)$. The correspondence

between ξ and ξ^* is linear. Let the ξ 's which give rise to bounded functionals form a subspace $\mathcal{N}$ having $\mathcal{M}$ as its least extension. Then we obtain in this way a linear operator $\Omega^* \xi = \xi^*$ so that $(\Omega \mu, \xi) = (\mu, \Omega^* \xi)$. We speak of Ω^* (on $\mathcal{N}$) as the adjoint operator to Ω . Now consider bounded linear operators on $\mathcal{M}$. Obviously $\mathcal{M}(\Omega^*) = \mathcal{M}$ and $M \Omega = M \Omega^*$. Further $(\Phi \Omega)^* = \Omega^* \Phi^*$, since $(\Phi \Omega \mu, \xi) = (\Omega \mu, \Phi^* \xi) = (\mu, \Omega^* \Phi^* \xi)$.

The operators that play a leading role in the theory are hermitian, i.e., $(\Omega \mu, \xi) = (\mu, \Omega \xi)$ for every pair μ, ξ in $\mathcal{M}(\Omega)$. A hermitian operator is always linear provided $\mathcal{M}(\Omega)$ is linear and has $\mathcal{M}$ as its least extension. For $(\Omega(a\mu + b\xi), \gamma) = a(\mu, \Omega \gamma) + b(\xi, \Omega \gamma) = a(\Omega \mu, \gamma) + b(\Omega \xi, \gamma)$ so that

$$(\Omega(a\mu + b\xi) - (a\Omega\mu + b\Omega\xi), \gamma) = 0$$

and hence $\Omega(a\mu + b\xi) = a\Omega\mu + b\Omega\xi$ as $\mathcal{M}$ is the extension of $\mathcal{M}(\Omega)$. Connected with a hermitian operator we define the number $\underline{M}_0$ to be the greatest lower bound of $(\Omega \mu, \mu)$ with $\|\mu\| \leq 1$. If the space does not consist of the zero vector alone, we define $\underline{M}$ to be the greatest lower bound of $(\Omega \mu, \mu)$ but with $\|\mu\| = 1$. Similar definitions apply to $\overline{M}_0, \overline{M}$. A hermitian operator is said to be positive or negative according as $(\Omega \mu, \mu)$ is positive or negative for every μ in $\mathcal{M}(\Omega)$.

THEOREM 3. If Ω is bounded and hermitian with $\mathcal{M}(\Omega)$ dense on $\mathcal{M}$, then $M \Omega$ is equal to the least upper bound of $(\Omega \mu, \mu)$ for $\|\mu\| \leq 1$, and

$$M \Omega = (1/2)(|\overline{M}_0| + |\underline{M}_0|) + (1/2)(|\overline{M}_0| - |\underline{M}_0|),$$

i.e., the greater of $|\underline{M}_0|, |\overline{M}_0|$. In case $\mathcal{M}$ has non-zero vectors, these equalities still hold when $\underline{M}_0, \overline{M}_0$ are replaced by $\underline{M}, \overline{M}$, and

the condition on the μ 's by $\|\mu\| = 1$.

To establish these equalities, first we observe that they are trivial when $\mathfrak{M}$ consists of a single zero element. Next, the inequality $|(\Omega\mu, \xi)| \leq \text{gr}(|\underline{M}_0|, |\overline{M}_0|)$ evidently holds if $(\Omega\mu, \xi) = 0$. Consider the case $(\Omega\mu, \xi) \neq 0$. We note with F. Riesz that

$$\begin{aligned} 2|(\Omega\mu, \xi)| + |(\Omega\xi, \mu)| &= |(\Omega(\mu + \xi), \mu + \xi) - (\Omega(\mu - \xi), \mu - \xi)| \\ &\leq \text{gr}(|\underline{M}_0|, |\overline{M}_0|) \{ \|\mu + \xi\|^2 + \|\mu - \xi\|^2 \} \\ &\leq 2 \text{gr}(|\underline{M}_0|, |\overline{M}_0|) \{ \|\mu\|^2 + \|\xi\|^2 \} \end{aligned}$$

Replace ξ by $a\xi$, $a \equiv (\Omega\mu, \xi)/|(\Omega\mu, \xi)|$. Then the right hand side remains unchanged, while in the left hand side each of the two terms becomes $|(\Omega\mu, \xi)|$ due to the special value of a . We have consequently $2|(\Omega\mu, \xi)| \leq \text{gr}(|\underline{M}_0|, |\overline{M}_0|) \{ \|\mu\|^2 + \|\xi\|^2 \}$ which reduces to $|(\Omega\mu, \xi)| \leq \text{gr}(|\underline{M}_0|, |\overline{M}_0|)$ for $\|\mu\|^2 \leq 1$, $\|\xi\|^2 \leq 1$, so that $M\Omega \leq \text{gr}(|\underline{M}_0|, |\overline{M}_0|)$, by a theorem proved above. On the other hand, $|(\Omega\mu, \mu)| \leq M\Omega$ for $\|\mu\| \leq 1$. Now for every set of real numbers $\{r\}$ $\overline{B}|r|$ is the greater of $|\overline{B}r|$ and $|\underline{B}r|$ so that for $\|\mu\| \leq 1$ we have $\overline{B}|(\Omega\mu, \mu)| = \text{gr}(|\underline{M}_0|, |\overline{M}_0|)$. This gives $\text{gr}(|\underline{M}_0|, |\overline{M}_0|) \leq M\Omega$ and the equalities $\overline{B}|(\Omega\mu, \mu)| = \text{gr}(|\underline{M}_0|, |\overline{M}_0|) = M\Omega$ are established. This argument still holds when we use $\overline{M}$, $\underline{M}$ instead of $\overline{M}_0$, $\underline{M}_0$.

If $\underline{M}\Omega = \overline{M}\Omega = s$ then $\Omega = sI$, where $\underline{M}\Omega \equiv \underline{M}$, $\overline{M}\Omega \equiv \overline{M}$, as before. For

$$\underline{M}\Omega(\mu, \mu) \leq (\Omega\mu, \mu) \leq \overline{M}\Omega(\mu, \mu),$$

and hence $(\Omega\mu, \mu) = s(\mu, \mu)$. Polarizing this equality we find $(\Omega\mu, \xi) = s(\mu, \xi)$. Thus $\Omega = sI$.

In his doctoral dissertation Y. K. Wong has recently generalized the interesting Hellinger-Toeplitz theorem to matrices in general analysis. By converting a given hermitian operator into a matrix we can easily show, using this theorem, that a hermitian operator defined on the entire space is necessarily bounded. We need not mention the details of proof beyond saying that the previous results concerning the Fourier situation (Chapter I), having the whole space as its basis, enter in an essential step. From this fact about hermitian operators we immediately derive a more general theorem which has its matricial instance already given in general analysis and its denumerable case proved by Stone in his book.

THEOREM 4. If a linear operator and its adjoint are both defined in the entire complete unitary space then they are bounded.

3. Bilinear functionals

A numerical-valued function $B(\mu, \xi)$, μ, ξ being any pair of elements in a unitary space $\mathcal{H}$ is called a bilinear functional in case

$$B(\mu, a\xi + b\eta) = B(\mu, \xi)a + B(\mu, \eta)b,$$

$$B(a\mu + b\xi, \eta) = aB(\mu, \eta) + bB(\xi, \eta).$$

Let $B_R(\mu)$ stand for $B(\mu, \mu)$ and put

$$E_m(\mu, \xi) \equiv B_R(\mu + v_m \xi) - B_R(\mu - v_m \xi),$$

where v_m are the units 1 or i , i or 1 , i , j , k in the number system $\mathcal{H}$. Then, by following the proofs given in general analysis we obtain the usual polarization formulas:

$$2B(\mu, \xi) + 2B(\xi, \mu) = E_1(\mu, \xi) \quad \text{for the reals,}$$

$$4B(\mu, \xi) = \sum_{m=1}^2 E_m(\mu, \xi) v_m \quad \text{for the complexes,}$$

$$12B(\mu, \xi) = \sum_{m=1}^4 [2E_m(\mu, \xi) + \bar{E}_m(\mu, \xi)] v_m \quad \text{for the quaternions.}$$

A bilinear functional B is said to be bounded when $|B(\mu, \xi)|$ is uniformly bounded for all $\|\mu\| \leq 1$, $\|\xi\| \leq 1$ and the least effective r in $|B(\mu, \xi)| \leq r \|\mu\| \|\xi\|$ is called the modulus of B , written as MB . By the XY-continuity of a bilinear functional we mean that for any $\mathcal{P}R$ and $\mathcal{Q}S$ -systems

$$B(\mu, \xi) = L_{pq} B(\mu_p, \xi_q)$$

where $\{\mu_p\}$ converge to μ in the X -sense and $\{\xi_q\}$ to ξ in the Y -sense with X, Y denoting either S or W . If the continuity holds only for normal denumerable sequences then the functional is said to have an X_0Y_0 -continuity.

Consider a bilinear functional with S_0S_0 -continuity.

Suppose that the functional is unbounded, then for every n there is a pair μ_n, ξ_n such that $\|\mu_n\| = 1 = \|\xi_n\|$ and $|B(\mu_n, \xi_n)| > n^2$. Now $\|\mu_n/\sqrt{n}\|$ and $\|\xi_n/\sqrt{n}\|$ both converge to zero and hence $\{\mu_n\}, \{\xi_n\}$ have 0 as their strong limit. It follows then

$$\lim_{m,n} B(\mu_m/\sqrt{m}, \xi_n/\sqrt{n}) = 0$$

so that $|B(\mu_m/\sqrt{m}, \xi_n/\sqrt{n})|$ are uniformly bounded. But $B(\mu_n/\sqrt{n}, \xi_n/\sqrt{n}) = (1/n)B(\mu_n, \xi_n)$. Consequently $|B(\mu_n/\sqrt{n}, \xi_n/\sqrt{n})| > n$. Thus the supposition cannot be justified and the functional is bounded. If, however, the functional is bounded, then $|B(\mu_p - \mu, \xi_q - \xi)| \leq MB \cdot \|\mu_p - \mu\| \cdot \|\xi_q - \xi\|$. Suppose that μ_p converge weakly to μ , then $\|\mu_p - \mu\|$ are uniformly bounded for $pRpo$. Thus, for a strongly convergent system $\{\xi_q\}$,

$$L_{pq}B(\mu_p, \xi_q) = B(\mu, \xi).$$

These considerations lead to a theorem on conditions for the boundedness of a bilinear functional.

THEOREM 5. For a bilinear functional, defined on a unitary space, to be bounded it is necessary and sufficient that it has one of the following six types of continuity:

- (1) W_0S_0 , (2) S_0W_0 , (3) S_0S_0 , (4) WS ,
 (5) SW , (6) SS .

Using the representation theorem for linear functionals (Chapter I) we extend it to bilinear functionals.

THEOREM 6. Let B be a bounded bilinear functional on a complete unitary space. Then there exists uniquely a linear operator Ω such that $B(\mu, \xi) = (\mu, \Omega\xi)$. Further $MB = M\Omega$.

4. Limit of operators

Let $\{\Omega_\chi\}$ be an $\mathcal{L}_R$ -system of operators in a unitary space $\mathcal{M}$. We say that the system converges weakly or strongly to an operator Ω as on $\mathcal{N}$ if $\mathcal{N}$ lies in the intersection of the subspaces $\mathcal{M}(\Omega_\chi)$ and if $\Omega_\chi\xi$ converges weakly or strongly to $\Omega\xi$ for every ξ in $\mathcal{N}$. The system converges modularly to Ω in case $M(\Omega_\chi - \Omega)$ converges to zero. Obviously, for each of the first two types of convergence the corresponding Cauchy sufficiency condition holds as in the numerical case, provided $\mathcal{M}$ is complete.

We note that if

$$\|M_{\chi_1\chi_2}(\Omega_{\chi_1} - \Omega_{\chi_2})\| = Z,$$

then

$$\|S_{\chi_1\chi_2}(\Omega_{\chi_1} - \Omega_{\chi_2})\| = Z$$

uniformly on the set $\mathcal{N}_1$ of all unitary vectors in $\mathcal{N}$. Let

$\mu_{\chi p} \equiv \Omega_{\chi} \zeta_p$, $\mathcal{P} \equiv \mathcal{N}_1$. Then

$$\lim_{\chi_1 \chi_2} \|\mu_{\chi_1 p} - \mu_{\chi_2 p}\| = 0 \quad \text{uniformly on } \mathcal{P}.$$

This gives, by the Cauchy sufficiency condition for uniformly convergent systems of vectors,¹ the existence of a limit μ_p and

$$\lim_{\chi} \|\mu_{\chi p} - \mu_p\| = 0 \quad \text{uniformly on } \mathcal{P}.$$

Hence, by virtue of the unicity of the strong limit,

$$\lim_{\chi} \|\Omega_{\chi} \zeta - \Omega \zeta\| = 0 \quad \text{uniformly on } \mathcal{N}_1.$$

From this it follows immediately that $L_{\chi} M(\Omega_{\chi} - \Omega) = 0$. Thus the Cauchy sufficiency condition also holds for modular convergence.

An easier consequence from the limit definition is that for a weakly convergent system of hermitian operators in a unitary space the limiting operator remains hermitian.

Consider two systems of permutable operators Ω_p and Ω_q on $\mathcal{N}$. If $\Phi \Omega$ has sense and if Φ_p , Ω_q are uniformly bounded, then the simultaneous limit $L_{pq} \Phi_p \Omega_q$ exists in the strong sense and is equal to $\Phi \Omega$, provided the initial systems are strongly convergent. We have a weak limit if one of the two systems converges strongly while the other weakly. These facts can be readily seen when we recall the equalities

$$\begin{aligned} \Phi_p \Omega_q - \Phi \Omega &= (\Phi_p \Omega_q - \Phi_p \Omega) + (\Phi_p \Omega - \Phi \Omega) = (\Phi_p \Omega_q - \Phi_p \Omega) + (\Omega \Phi_p - \Omega \Phi) \\ &= (\Phi_p \Omega_q - \Phi \Omega_q) + (\Phi \Omega_q - \Phi \Omega) = (\Omega_q \Phi_p - \Omega_q \Phi) + (\Phi \Omega_q - \Phi \Omega). \end{aligned}$$

The idea of operatorial limit can now be utilized to prove a useful theorem.²

¹The proof of this fact runs just as in the numerical case.

²See Riesz, Szeged Acta, p. 33.

THEOREM 7. Let Ω, Φ be two positive bounded hermitian operators on a unitary space $\mathcal{M}$. If they are permutable on $\mathcal{M}$ their product $\Omega\Phi$ is also positive hermitian.

For simplicity suppose that $M\Omega \leq 1$ and hence $I - \Omega$ is positive. Define $\Omega_1 \equiv \Omega$, $\Omega_{n+1} \equiv \Omega_n - \Omega_n^2$.

$$\Omega_{n+1} = \Omega_n(I - \Omega_n) = \Omega_n(I - \Omega_n)(\Omega_n + I - \Omega_n) = \Omega_n^2(I - \Omega_n) + \Omega_n(I - \Omega_n)^2$$

We proceed to show that $\Omega_m, I - \Omega_m$ are positive for every m .

To make an induction argument, suppose that $\Omega_n, I - \Omega_n$ are positive. So are $\Omega_n(I - \Omega_n)\Omega_n$ and $(I - \Omega_n)\Omega_n(I - \Omega_n)$.

Being a sum of two positive operators Ω_{n+1} is evidently positive.

On the other hand

$$I - \Omega_{n+1} = I - (\Omega_n - \Omega_n^2) = (I - \Omega_n) + \Omega_n^2.$$

Consequently by induction Ω_m and $I - \Omega_m$ are both positive.

Next, we wish to show that $\Omega = \Omega_1^2 + \dots + \Omega_m^2 + \Omega_{m+1}$ for every m . This formula holds for $m = 1$, since

$$\Omega = \Omega^2 + (\Omega - \Omega^2) = \Omega_1^2 + \Omega_2.$$

Further, if $\Omega = \Omega_1^2 + \dots + \Omega_{n-1}^2 + \Omega_n$, then $\Omega = \Omega_1^2 + \dots + \Omega_{n-1}^2 + (\Omega_n^2 + \Omega_{n+1})$ from the definition of Ω_n . Hence

$\Omega - (\Omega_1^2 + \dots + \Omega_m^2) = \Omega_{m+1}$ is positive, and $(\Omega_1^2\mu, \mu) + \dots + (\Omega_m^2\mu, \mu) \leq (\Omega\mu, \mu)$. It follows then

$$\sum_{m=1}^{\infty} (\Omega_m^2\mu, \mu) \leq (\Omega\mu, \mu)$$

which in turn implies $L_m \|\Omega_m\mu\|^2 = L_m(\Omega_m^2\mu, \mu) = 0$. As

$\Omega = \sum_{m=1}^n \Omega_m^2 + \Omega_{n+1}$ for every n ,

$$(\Omega\mu, \mu) = \lim_{n \rightarrow \infty} \sum_{m=1}^n (\Omega_m^2\mu, \mu) + \lim_{n \rightarrow \infty} (\Omega_{n+1}\mu, \mu) = \sum_{m=1}^{\infty} (\Omega_m^2\mu, \mu).$$

Polarizing the terms $(\Omega\mu, \mu)$, $(\Omega_m^2\mu, \mu)$ we obtain

$$(\Omega\mu, \xi) = \sum_{m=1}^{\infty} (\Omega_m^2\mu, \xi).$$

Thus we have

$$\bigcup_n \sum_{m=1}^n \Omega_m^2 = \Omega.$$

It follows that

$$\Omega\Phi = \left(\sum_{m=1}^{\infty} \Omega_m^2 \right) \Phi = \sum_{m=1}^{\infty} (\Omega_m^2 \Phi) = \sum_{m=1}^{\infty} (\Omega_m \Phi \Omega_m).$$

So, $\Omega\Phi$ is a sum of infinitely many positive operators $\Omega_m \Phi \Omega_m$ and consequently is positive itself.

5. Operatorial-valued measure of sets

In this and the following sections of this chapter the spaces used are complete unitary.

Let F_t be an operatorial-valued function of t ($-\infty < t < +\infty$) where the operators are all bounded on $\mathcal{M}$ but need not be linear. Then we say that F_t is of limited variation in the operatorial sense in case there exists a positive number r such that the total variation of $(\xi, F_t\mu)$ over any interval on the t -line is not greater than $r \|\mu\| \cdot \|\xi\|$. A set S is F_t -measurable if and only if S is measurable with respect to $(\xi, F_t\mu)$ (or its numerical components in case of complex and quaternionic numbers) for every pair of elements μ, ξ . We call an operator $\Phi(S)$ on $\mathcal{M}$ an operatorial F_t -measure of S when S is F_t -measurable and has $(\xi, \Phi(S)\mu)$ as its measure relative to $(\xi, F_t\mu)$.

THEOREM 8. (Existence of operatorial measure). The F_t -measure always exists uniquely and is bounded for every F_t -measurable set.

Consider a definite μ . As in the ordinary theory, an F_t -measurable S has its numerical measure $M(\xi; S)$ relative to $(\xi, F_t\mu)$ for each given ξ . Now it can be shown that for $\gamma(t) \equiv a\alpha(t) + b\beta(t)$ the $\gamma(t)$ -measure of an $\alpha(t)$ - and $\beta(t)$ -measurable set S is equal to the corresponding sum of the $\alpha(t)$ - and $\beta(t)$ -measures of S . Hence

$$M(a\xi + b\gamma; S) = aM(\xi; S) + bM(\gamma; S).$$

But we have also

$$|M(\xi; S)| \leq (r \|\mu\|) \|\xi\|,$$

since F_t is of limited variation in the operatorial sense. Thus $M(\xi; S)$ is a bounded linear functional. So, by the representation theorem in Chapter I there exists uniquely a μ^* such that $(\xi, \mu^*) = M(\xi; S)$. Define an operator $\Phi(S)$ by $\Phi(S)\mu = \mu^*$. Then $(\xi, \Phi(S)\mu)$ is the measure of S with respect to $(\xi, F_t\mu)$. Further, $\Phi(S)$ is bounded since $|\xi, \Phi(S)\mu| \leq r \|\xi\| \cdot \|\mu\|$.

We note that the numerical measure $M(\xi; S)$ for a given pair ξ, μ is completely additive and does not exceed $r \|\xi\| \cdot \|\mu\|$. This leads to the following

THEOREM 9. (Complete additivity of operatorial measure).

Let $\{S_m\}$ be a sequence of disjoint F_t -measurable sets on the t -line. Then

$$M\Phi\left(\sum_{m=1}^{\infty} S_m\right), \quad M\Phi\left(\sum_{m=1}^n S_m\right)$$

are all uniformly bounded, and

$$\Phi\left(\sum_{m=1}^{\infty} S_m\right) = \lim_{n \rightarrow \infty} \Phi\left(\sum_{m=1}^n S_m\right).$$

6. Projection operators

Previously we pointed out the projection character of the Fourier correspondence. Associated with any set of points there is a hermitian operator P such that $P^2 = P$. Consider in general a closed hermitian operator P defined on a complete linear subspace, satisfying the relations

$$P\mathcal{M}(P) \subset \mathcal{M}(P), \quad P^2 = P.$$

Such an operator we call a projection operator.

Let $P\mathcal{M}(P) \equiv \mathcal{N}$. Then for the projection operator $P_{\mathcal{N}}$ we have

$$\begin{aligned} \mathcal{N} &\equiv [\xi], \quad \epsilon(\pi, \pi) \equiv (\xi_1, \xi_2) / \xi_1 \xi_2, \quad P_{\mathcal{N}} \mu \equiv K(\mu, \pi) \pi, \\ P_{\mathcal{N}} \xi &= \xi, \quad P\xi = P(P\xi) = P\xi = \xi, \end{aligned}$$

where ξ is an element corresponding to ξ via P , $\xi = P\xi$. Hence $P_{\mathcal{N}} = P$ as on $\mathcal{N}$ and $P(\xi - P\xi) = 0$. But

$$(\xi - P\xi, \pi) = (\xi - P\xi, P\pi) = (P\xi - P\xi, \pi) = 0,$$

so that $\xi - P\xi$ is orthogonal to $\mathcal{N}$ and consequently $P_{\mathcal{N}}(\xi - P\xi) = 0$. Thus $P(\xi - P\xi) = P_{\mathcal{N}}(\xi - P\xi) = 0$. From $PP\xi = P\xi = P_{\mathcal{N}}P\xi$ we have $P\xi = P_{\mathcal{N}}\xi$. Further, $\mathcal{M}(P) = \mathcal{M}$, $P = P_{\mathcal{N}}$ as on $\mathcal{M}$, since

$$\lim_n P \xi_n = P \lim_n \xi_n.$$

These facts may be summarized in

THEOREM 10. There exists a one-to-one correspondence between a complete unitary subspace $\mathcal{N}$ and a projection operator P such that $\mathcal{N} = P\mathcal{M}$, $P = P_{\mathcal{N}}$ on $\mathcal{M}$. A projection operator is always bounded since $P_{\mathcal{N}}$ is so.

To state the following theorems we need some abbreviations. We write $\mathcal{M}_{1+2}$ for the complete linear extension of $\mathcal{M}_1 + \mathcal{M}_2$, $0\mathcal{M}$ for the orthogonal complement of $\mathcal{M}$ in $\mathcal{M}$, $0_1\mathcal{M}$ for that in $\mathcal{M}_1$ and so on. Further, P_{1+2} corresponds to $\mathcal{M}_{1+2}$, $P_{1 \times 2}$ to $\mathcal{M}_1 \times \mathcal{M}_2$.

THEOREM 11. Let P_1, P_2 be projection operators and $\mathcal{M}_1, \mathcal{M}_2$ their corresponding complete unitary subspaces. Then $P_1 + P_2$ is a projection operator if and only if one of the following three equivalent conditions is satisfied:

$$P_1 P_2 = Z, \quad P_2 P_1 = Z, \quad \mathcal{M}_1 \cap \mathcal{M}_2 = \{0\}.$$

And in that case $P_1 + P_2 = P_{1+2}$, $\mathcal{M}_{1+2}$ being the complete linear extension of $\mathcal{M}_1 + \mathcal{M}_2$. In order that $P_1 - P_2$ be a projection operator it is necessary and sufficient that one of the following equivalent conditions holds:

$$P_1 P_2 = P_2, \quad P_2 P_1 = P_2, \quad \mathcal{M}_1 \supset \mathcal{M}_2, \quad \|P_1 \mu\| \geq \|P_2 \mu\|.$$

If so, $P_1 - P_2 = P_{\mathcal{M}}$, $\mathcal{M}$ being the orthogonal complement of $\mathcal{M}_2$ within $\mathcal{M}_1$.

Since $(P_1 + P_2)^2 = P_1 + P_2$ it follows that $P_1 P_2 + P_2 P_1 = Z$. Multiplying this equality by P_1 we find $P_1 P_2 + P_1 P_2 P_1 = Z$. Again multiply the new equality by P_1 on the right to obtain $2P_1 P_2 P_1 = Z$. Thus,

$$Z = P_1 P_2 + P_1 P_2 P_1 = P_1 P_2 + Z = P_1 P_2,$$

$$Z = Z^* = (P_1 P_2)^* = P_2 P_1.$$

If $P_1 P_2 = Z$ then

$$\begin{aligned} (\mu_1, \mu_2) &= (P_1 \mu_1, P_2 \mu_2) = (\mu_1, P_1 P_2 \mu_2) \\ &= (\mu_1, 0) = 0, \end{aligned}$$

where μ_1, μ_2 lie in $\mathcal{M}_1, \mathcal{M}_2$ respectively. This shows that $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal. Let ξ be a point in $\mathcal{M}_{1+2}$. Then $\xi = \mu_1 + \mu_2$, and

$$\begin{aligned} (P_1 + P_2)\xi &= (P_1 + P_2)(\mu_1 + \mu_2) = P_1\mu_1 + P_2\mu_1 + P_1\mu_2 + P_2\mu_2 \\ &= \mu_1 + 0 + 0 + \mu_2 = \xi. \end{aligned}$$

As $P_1 + P_2$ is bounded we have $(P_1 + P_2)\zeta = \zeta$ for every ζ in the complete linear extension $\mathcal{M}_{1+2}$. Thus $\mathcal{M}_{1+2} = (P_1 + P_2)\mathcal{M}_{1+2}$ and $P_1 + P_2 = P_{1+2}$ as on $\mathcal{M}_{1+2}$. Now consider $\mathcal{N}$, the orthogonal space to $\mathcal{M}_{1+2}$. Because $\mathcal{N}$ is orthogonal to $\mathcal{M}_1$ and $\mathcal{M}_2$, $P_1\eta = 0$, $P_2\eta = 0$ for every η of $\mathcal{N}$. According to the unique decomposition of $\mathcal{M}$, $\mu = \zeta + \eta$. Consequently $(P_1 + P_2)\mu = (P_1 + P_2)(\zeta + \eta) = \zeta + P_1\eta + P_2\eta = \zeta + 0 = P_{1+2}\mu$.

To establish the conditions on $P_1 - P_2$ we observe that P is a projection operator if and only if $I - P$ is so. Let $P_1 - P_2$ be a projection operator. Then $I - (P_1 - P_2) = (I - P_1) + P_2$ is a projection operator. By the conditions on the sum of two projection operators we see immediately that $(I - P_1)P_2 = 0$ and hence $P_1P_2 = P_2$, $P_2P_1 = P_2$. If ξ is orthogonal to $\mathcal{M}_1$ then $P_1\xi = 0$ and $P_2\xi = P_2P_1\xi = 0$. It follows that ξ is orthogonal to $\mathcal{M}_2$ also and $\mathcal{M}_1$ contains $\mathcal{M}_2$. Since $\mathcal{M}_1 \supset \mathcal{M}_2$, $\mathcal{M}_1$ is the complete linear extension of $\mathcal{N} + \mathcal{M}_2$, where $\mathcal{N}$ and $\mathcal{M}_2$ are orthogonal. Then by what has been proved in the last paragraph regarding the projection operators of orthogonal spaces,

$$P_1 = P_{\mathcal{N}} + P_2.$$

Consequently, $P_1 - P_2$ is a projection operator of $\mathcal{N}$. When $\mathcal{M}_1$ contains $\mathcal{M}_2$,

$$\|P_2\mu\| = \|P_2P_1\mu\| \leq MP_2\|P_1\mu\| \leq \|P_1\mu\|.$$

Conversely, suppose that $\|P_2 \mu\| \leq \|P_1 \mu\|$, and let μ_2 be in $\mathcal{M}_2$. Then

$$\begin{aligned} \|\mu_2\| &= \|P_2 \mu_2\| \leq \|P_1 \mu_2\|, \quad (\mu_2, \mu_2) = \|\mu_2\|^2 \leq \|P_1 \mu_2\|^2 = (\mu_2, P_1 \mu_2), \\ \|(I - P_1) \mu_2\|^2 &= (\mu_2, (I - P_1) \mu_2) \leq 0, \quad (I - P_1) \mu_2 = 0, \end{aligned}$$

so that $P_1 \mu_2 = \mu_2$ and hence μ_2 is in $\mathcal{M}_1$. Thus $\mathcal{M}_2$ lies in $\mathcal{M}_1$.

The question: When can $P_1 P_2$ be a projection operator? seems more complicated, but we have the following answer first given for matrices in general analysis.

THEOREM 12. As before, let P_1, P_2 be projection operators with their corresponding complete linear spaces $\mathcal{M}_1, \mathcal{M}_2$. Then $P_1 P_2$ is a projection operator if and only if one of the following equivalent conditions is satisfied:

- (1) $P_1 P_2 = P_2 P_1$, (2) $P_1 P_2 = P_{1 \times 2}$,
- (3) $P_{1+2} = P_1 + P_2 - P_{1 \times 2}$, (4) $P_{1+2} = P_1 + P_2 - P_1 P_2$,
- (5) $O_{1+2} \mathcal{M}_1 = O_2 \mathcal{M}_{1 \times 2}$, (6) $O_2 \mathcal{M}_{1 \times 2} = O_2 \mathcal{M}_1$,
- (7) $O_{1+2} \mathcal{M}_1 = O_2 \mathcal{M}_1$, (8) $O_1 \mathcal{M}_{1 \times 2}$ and $O_2 \mathcal{M}_{1 \times 2}$ are orthogonal,
- (9) $O_{1+2} \mathcal{M}_1$ and $O_{1+2} \mathcal{M}_2$ are orthogonal.

Five more conditions can be obtained by parity with (2), (4), (5), (6), (7). It should be quite obvious that $P_1 P_2$ has projection character when and only when P_1, P_2 are permutable. For proof we use the schemes

$$(1) \rightarrow (2) \rightarrow (6) \rightarrow (8) \rightarrow (5) \rightarrow (7) \rightarrow (9) \rightarrow (4) \rightarrow (1),$$

$$(2)(4) \rightarrow (3) \rightarrow (1).$$

(1) $\rightarrow$ (2): As $\mathcal{M}_{1 \times 2} \subset \mathcal{M}_1, \mathcal{M}_{1 \times 2} \subset \mathcal{M}_2$ we have by the preceding theorem

$$P_{1 \times 2} P_1 = P_{1 \times 2} = P_{1 \times 2} P_2, \quad P_{1 \times 2} (P_1 P_2) = P_{1 \times 2}.$$

On the other hand, since $P_1P_2 = P_2P_1$, the product is evidently a projection operator. Now $\mathcal{N} \equiv (P_1P_2)\mathcal{M} \subset \mathcal{M}_{1 \times 2}$; so it follows again from the preceding theorem that $P_{1 \times 2}P\mathcal{N} = P\mathcal{N}$. But $P\mathcal{N} = P_1P_2$ on account of the unicity theorem regarding projections. Hence combining results we obtain

$$P_1P_2 = P_{1 \times 2}(P_1P_2) = P_{1 \times 2}.$$

For (2) $\rightarrow$ (6) let us write $\mathcal{N}$ for $O_2\mathcal{M}_{1 \times 2}$, $\mathcal{Q}$ for $P\mathcal{N}$. Then by the orthogonal property of $\mathcal{N}$, $QP_{1 \times 2} = Z$. Now $\mathcal{N} \subset \mathcal{M}_2$ so that $\mathcal{Q} = QP_2$. Hence from (2)

$$QP_1 = (QP_2)P_1 = Q(P_2P_1) = Q(P_1P_2) = QP_{1 \times 2} = Z.$$

Consequently $\mathcal{N}$, $\mathcal{M}_1$ are orthogonal. Then $O_2\mathcal{M}_{1 \times 2} \subset O_2\mathcal{M}_1$. On the other hand

$$O_2(\mathcal{M}_1 \times \mathcal{M}_2) = \mathcal{M}_2 \times O(\mathcal{M}_1 \times \mathcal{M}_2) \supset \mathcal{M}_2 \times O\mathcal{M}_1.$$

Thus we have the equality in (6).

(6) $\rightarrow$ (8): From (6), $O_1\mathcal{M}_{1 \times 2} \subset O\mathcal{M}_1$. Now, for $O\mathcal{R}$ to contain $\mathcal{N}$ it is necessary and sufficient that $\mathcal{R}$ lies in $O\mathcal{N}$, the subspaces $\mathcal{R}$, $\mathcal{N}$ being linear, complete. So in the present case we get $OO_2\mathcal{M}_{1 \times 2} \supset \mathcal{M}_1$, since $\mathcal{M}_2$ is linear and complete. By orthogonal decomposition,

$$\mathcal{M}_1 \supset \mathcal{M}_{1 \times 2} + O_1\mathcal{M}_{1 \times 2}, \quad OO_2\mathcal{M}_{1 \times 2} \supset \mathcal{M}_{1 \times 2} + O_1\mathcal{M}_{1 \times 2}.$$

Thus $OO_2\mathcal{M}_{1 \times 2} \supset O_1\mathcal{M}_{1 \times 2}$ and consequently $O_2\mathcal{M}_{1 \times 2}$ and $O_1\mathcal{M}_{1 \times 2}$ are orthogonal.

To prove (8) $\rightarrow$ (5)¹ we first show that $O_2\mathcal{M}_{1 \times 2} \subset O_{1+2}\mathcal{M}_1$. For that purpose we observe that

¹Here we use the proof given in general analysis.

$$0_2 \mathcal{M}_{1 \times 2} \subset \mathcal{M}_2, \quad 0_2 \mathcal{M}_{1 \times 2} \subset 0 \mathcal{M}_{1 \times 2}.$$

According to (8),

$$0_2 \mathcal{M}_{1 \times 2} \subset 00_1 \mathcal{M}_{1 \times 2}, \quad 0_2 \mathcal{M}_{1 \times 2} \subset 00_1 \mathcal{M}_{1 \times 2} \times 0 \mathcal{M}_{1 \times 2}.$$

With the aid of a general equality regarding two complete linear subspaces $\mathcal{R}, \mathcal{N}$, $0 \mathcal{R} \times 0 \mathcal{N} = 0(\mathcal{R} + \mathcal{N})$, we derive the following relations, by putting

$$\mathcal{R} \equiv 0_1 \mathcal{M}_{1 \times 2}, \quad \mathcal{N} \equiv \mathcal{M}_{1 \times 2},$$

$$0_2 \mathcal{M}_{1 \times 2} \subset 00_1 \mathcal{M}_{1 \times 2} \times 0 \mathcal{M}_{1 \times 2} = 0[0_1 \mathcal{M}_{1 \times 2} + \mathcal{M}_{1 \times 2}] = 0 \mathcal{M}_1.$$

Whence $0_2 \mathcal{M}_{1 \times 2} \subset 0_2 \mathcal{M}_1$. But $0_2 \mathcal{M}_1 \subset 0_{1+2} \mathcal{M}_1$ so that $0_2 \mathcal{M}_{1 \times 2} \subset 0_{1+2} \mathcal{M}_1$. To proceed further let us set $\mathcal{M}_3 = 0_{1+2} \mathcal{M}_1 \times 00_2 \mathcal{M}_{1 \times 2}$. Then

$$\mathcal{M}_3 \subset 00_2 \mathcal{M}_{1 \times 2}, \quad \mathcal{M}_3 \subset 0_{1+2} \mathcal{M}_1 = \mathcal{M}_{1+2} \times 0 \mathcal{M}_1 \subset 0 \mathcal{M}_1 \subset 0 \mathcal{M}_{1 \times 2}.$$

These show that

$$\mathcal{M}_3 \subset (00_2 \mathcal{M}_{1 \times 2}) \times 0 \mathcal{M}_{1 \times 2} = 0[0_2 \mathcal{M}_{1 \times 2} + \mathcal{M}_{1 \times 2}] = 0 \mathcal{M}_2.$$

Whence we must have

$$\mathcal{M}_3 \subset (0 \mathcal{M}_1) \times 0 \mathcal{M}_2 = 0 \mathcal{M}_{1+2},$$

on account of a general equality quoted above. But $\mathcal{M}_3 \subset \mathcal{M}_{1+2}$ as $\mathcal{M}_3 \subset \mathcal{M}_{1+2} \times 0 \mathcal{M}_1$. Thus $\mathcal{M}_3 \subset 0 \mathcal{M}_{1+2} \times \mathcal{M}_{1+2}$, so that $\mathcal{M}_3$ consists of a single zero-vector. Now let $\mathcal{M}_4$ stand for $0_{1+2} \mathcal{M}_1$, $\mathcal{M}_5$ for $0_2 \mathcal{M}_{1 \times 2}$. Then, $\mathcal{M}_5 \subset \mathcal{M}_4$ as proved above, and

$$0_4 \mathcal{M}_5 = \mathcal{M}_4 \times 00_2 \mathcal{M}_{1 \times 2} = (0_{1+2} \mathcal{M}_1) \times 00_2 \mathcal{M}_{1 \times 2} = \mathcal{M}_3 = 0.$$

It follows that $\mathcal{M}_5$ is dense on $\mathcal{M}_4$. This means simply that $\mathcal{M}_4 = \mathcal{M}_5$ or $0_{1+2} \mathcal{M}_1 = 0_2 \mathcal{M}_{1 \times 2}$, since both are linear and complete.

(5) $\rightarrow$ (7): Using the obvious equalities

$$O_2 \pi_{1 \times 2} = \pi_2 \times O \pi_{1 \times 2}, \quad O_{1+2} \pi_1 = \pi_{1+2} \times O \pi_1$$

we find readily

$$\begin{aligned} (O_2 \pi_{1 \times 2}) \times O_{1+2} \pi_1 &= \pi_2 \times O \pi_{1 \times 2} \times \pi_{1+2} \times O \pi_1 \\ &= (\pi_2 \times \pi_{1+2}) \times (O \pi_1) \times O \pi_{1 \times 2} = \pi_2 \times O \pi_1. \end{aligned}$$

But by (5), $O_2 \pi_{1 \times 2} = O_{1+2} \pi_1$, and so

$$\pi_2 \times O \pi_1 = (O_2 \pi_{1 \times 2}) \times O_{1+2} \pi_1 = O_{1+2} \pi_1.$$

(7) $\rightarrow$ (9): By (7), $O_{1+2} \pi_1 = \pi_2 \times O \pi_1$, so that $O_{1+2} \pi_1 \subset \pi_2$. This gives

$$O(O_{1+2} \pi_1) \supset O_{1+2}(O_{1+2} \pi_1) \supset O_{1+2} \pi_2.$$

Obviously then $O_{1+2} \pi_1, O_{1+2} \pi_2$ are orthogonal.

(9) $\rightarrow$ (4): For every pair ξ, ζ in π_{1+2} , $\xi - P_1 \xi$ is in π_{1+2} and orthogonal to π_1 , $\zeta - P_2 \zeta$ lies in $O_{1+2} \pi_2$. Then by (9)

$$\begin{aligned} 0 &= (\xi - P_1 \xi, \zeta - P_2 \zeta) = (\xi, \zeta) - (\xi, P_1 \zeta) - (\xi, P_2 \zeta) + \\ &\quad + (\xi, P_1 P_2 \zeta) = (\xi, P_{1+2} \zeta) - (\xi, P_1 \zeta) - (\xi, P_2 \zeta) + \\ &\quad + (\xi, P_1 P_2 \zeta). \end{aligned}$$

On the other hand,

$$(\gamma, (P_{1+2} - P_1 - P_2 + P_1 P_2) \nu) = 0,$$

where γ, ν are elements of $O \pi_{1+2}$. Now for elements μ, φ in π we have

$$\mu = \mu_+ + \mu_0, \quad \varphi = \varphi_+ + \varphi_0$$

with μ_+, φ_+ in π_{1+2} , μ_0, φ_0 in $O \pi_{1+2}$, and

$$\begin{aligned} (\varphi, (P_{1+2} - P_1 - P_2 + P_1 P_2)\mu) &= (\varphi, (P_{1+2} - P_1 - P_2 + P_1 P_2)\mu_+) \\ &= (\varphi_+, (P_{1+2} - P_1 - P_2 + P_1 P_2)\mu_+) = 0. \end{aligned}$$

It follows that $P_{1+2} - P_1 - P_2 + P_1 P_2 = Z$, since, for every μ

$$\begin{aligned} \|(P_{1+2} - P_1 - P_2 + P_1 P_2)\mu\|^2 &= ((P_{1+2} - P_1 - P_2 + P_1 P_2)\mu, \\ &\quad (P_{1+2} - P_1 - P_2 + P_1 P_2)\mu) = 0, \end{aligned}$$

on putting $\varphi = (P_{1+2} - P_1 - P_2 + P_1 P_2)\mu$.

(4) $\rightarrow$ (1): According to (4), $P_1 P_2 = P_1 + P_2 - P_{1+2}$, so that

$$P_2 P_1 = (P_1 P_2)^* = (P_1 + P_2 - P_{1+2})^* = P_1 + P_2 - P_{1+2}.$$

Whence

$$P_2 P_1 = P_1 + P_2 - P_{1+2} = P_1 P_2.$$

It remains to show (2)(4) $\rightarrow$ (3) $\rightarrow$ (1). We observe first that (2) and (4) imply (3) plainly. Now, because of (3), $P_2 = P_{1+2} + P_{1 \times 2} - P_1$. This leads at once to (1), for

$$\begin{aligned} P_1 P_2 &= P_1(P_{1+2} + P_{1 \times 2} - P_1) = P_1 + P_{1 \times 2} - P_1 = P_{1 \times 2} \\ &= (P_{1 \times 2})^* = (P_1 P_2)^* = P_2 P_1. \end{aligned}$$

Hitherto we have considered only permutable cases and then obtained a formula for $P_{1 \times 2}$. In general analysis expressions are found for $P_{1 \times 2}$ (there a matrix) even when the matrices in question do not permute. The next theorem furnishes equalities for $P_{1 \times 2}$ and P_{1+2} . In these equalities the sequences involved are shown to be strongly convergent.

THEOREM 13. Let P_1, P_2 be two arbitrary projection operators with their corresponding spaces $\mathcal{M}_1, \mathcal{M}_2$, and $P_{1 \times 2}, P_{1+2}$ be the projection operators of the intersection space and the sum

space of $\mathcal{M}_1, \mathcal{M}_2$. Then the following formulas hold:

$$P_{1 \times 2} = \lim_{n \rightarrow \infty} (P_1 P_2)^n P_1 = \lim_{n \rightarrow \infty} (P_2 P_1)^n P_2 = \lim_{n \rightarrow \infty} (P_1 P_2)^n = \lim_{n \rightarrow \infty} (P_2 P_1)^n,$$

$$P_{1+2} = I - \lim_{n \rightarrow \infty} (I - P_1 - P_2 + P_1 P_2)^n = I - \lim_{n \rightarrow \infty} (I - P_1 - P_2 + P_2 P_1)^n.$$

In the proof let $\pi_n \equiv [(P_1 P_2)(P_2 P_1)]^n$. Now $I - \pi_1$ and π_n are permutable positive hermitian operators so that $\pi_n(I - \pi_1)$ is positive hermitian, in virtue of a previous theorem. Accordingly $(\pi_n \mu, \mu)$ is a monotone decreasing sequence for every μ and the limit $L_n(\pi_n \mu, \mu)$ exists. Polarizing these terms and observing that $\|\pi_n \mu\| \leq \|\mu\|$, we have

$$\lim_{n \rightarrow \infty} \pi_n \equiv \pi_*.$$

From the equalities

$$\pi_* \pi_n = (\lim_{m \rightarrow \infty} \pi_m) \pi_n = \lim_{m \rightarrow \infty} (\pi_m \pi_n) = \lim_{m \rightarrow \infty} \pi_{m+n} = \pi_*$$

we see that

$$\pi_*^2 = \pi_* (\lim_{n \rightarrow \infty} \pi_n) = \lim_{n \rightarrow \infty} (\pi_* \pi_n) = \pi_*.$$

Thus π_* is a projection operator. Further, $\pi_n^2 = \pi_{2n}$, and hence

$$\lim_{n \rightarrow \infty} \pi_n^2 = \pi_* = \pi_*^2.$$

Consequently, $L_n \|\pi_n \mu\|^2 = \|\pi_* \mu\|^2$ which with the weak convergence shows that $\{\pi_n\}$ converges strongly to π_* . To identify π_* with $P_{1 \times 2}$ we note first

$$P_1 \pi_* = \pi_* = \pi_* P_1.$$

On the other hand,

$$(P_2 \pi_n - \pi_n)^* (P_2 \pi_n - \pi_n) = (\pi_n P_2 - \pi_n) (P_2 \pi_n - \pi_n)$$

$$= \pi_n^2 - \pi_n P_2 \pi_n = \pi_{2n} - (P_1 P_2)^n P_1 P_2 (P_1 P_2)^n P_1 = \pi_{2n} - \pi_{2n+1},$$

since $\pi_n = (P_1 P_2)^n P_1$. It follows then that

$$(P_2 \pi^* - \pi^*)^* (P_2 \pi^* - \pi^*) = Z,$$

and we obtain

$$P_2 \pi^* = \pi^* = \pi^* P_2.$$

As P_1 , P_2 , π^* are projection operators, these equalities imply that π^* which corresponds to π^* lies in $\mathcal{M}_{1 \times 2}$ and that

$$P_{1 \times 2} \pi^* = \pi^* = \pi^* P_{1 \times 2}.$$

But $P_{1 \times 2} \pi_n = P_{1 \times 2}$ since $P_{1 \times 2} P_1 = P_{1 \times 2} = P_{1 \times 2} P_2$. Consequently,

$$P_{1 \times 2} \pi^* = P_{1 \times 2} = \pi^* = \pi^* P_{1 \times 2},$$

and, because of a previous equality, $\pi^* = P_{1 \times 2}$. From this the other equalities for $P_{1 \times 2}$ are obtained either as corollaries or by parity. To establish the formulas for P_{1+2} we need only recall that $I - P_1$ corresponds to $0\mathcal{M}_1$, $I - P_2$ to $0\mathcal{M}_2$, P_{1+2} to $0[(0\mathcal{M}_1) \times 0\mathcal{M}_2]$.

THEOREM 14. Let $\{P_\chi\}$ be a monotone $\mathcal{L}$ R-system of projection operators. Then there exists uniquely a projection operator P such that

$$P = \bigvee_{\chi} P_\chi.$$

Moreover, when the system is non-decreasing P is the projection operator of the sum space of spaces corresponding to the initial operators; it is the projection operator of the common intersection space if the system is non-increasing. Here the sense of monotonicity may be taken either analytically in terms of $\|P_\chi \mu\|$, or geometrically in terms of the spaces $\mathcal{M}_\chi$ corresponding to P_χ .

We begin the proof with a remark that the two senses mentioned are equivalent by virtue of a theorem concerning $P_1 - P_2$. Let us consider the case of a non-decreasing system. Then, for $\chi_2 R \chi_1$ the difference $P_{\chi_2} - P_{\chi_1}$ is a projection operator, and

$$\begin{aligned} 0 \leq ((P_{\chi_2} - P_{\chi_1})\mu, \mu) &= (P_{\chi_2}\mu, \mu) - (P_{\chi_1}\mu, \mu) \\ &= \|P_{\chi_2}\mu\|^2 - \|P_{\chi_1}\mu\|^2. \end{aligned}$$

$\|P_{\chi}\mu\| \leq \|\mu\|$ and $\|P_{\chi}\mu\|^2$ is a bounded monotone system, so that the limit $\lim_{\chi} \|P_{\chi}\mu\|^2$ exists. Now

$$\|P_{\chi_2}\mu - P_{\chi_1}\mu\|^2 = ((P_{\chi_2} - P_{\chi_1})\mu, \mu) = \|P_{\chi_2}\mu\|^2 - \|P_{\chi_1}\mu\|^2.$$

Since

$$\lim_{\chi_1 \chi_2} (\|P_{\chi_2}\mu\|^2 - \|P_{\chi_1}\mu\|^2) = 0,$$

we have

$$\lim_{\chi_1 \chi_2} \|P_{\chi_2}\mu - P_{\chi_1}\mu\|^2 = 0.$$

which by the general Cauchy sufficiency condition shows that there exists a μ^* such that

$$\lim_{\chi} P_{\chi}\mu = \mu^*.$$

Define an operator P by $P\mu \equiv \mu^*$. Hence

$$\lim_{\chi} P_{\chi}\mu = P\mu, \quad \lim_{\chi} (P_{\chi}\mu, \mu) = (P\mu, \mu), \quad MP \leq 1.$$

In fact

$$(P\mu, \frac{1}{2}\mu) = \lim_{\chi} (P_{\chi}\mu, \frac{1}{2}\mu) = \lim_{\chi} (\mu, P_{\chi}\frac{1}{2}\mu) = (\mu, P\frac{1}{2}\mu).$$

So, P is a bounded hermitian operator. Further, since P, P_{χ} are permutable,

$$P^2 = (\lim_{\chi} P_{\chi})(\lim_{\chi} P_{\chi}) = \lim_{\chi} P_{\chi}^2 = \lim_{\chi} P_{\chi} = P;$$

this shows its projection character. Let $\mathcal{M}_+$ be the sum space of spaces $\mathcal{M}_\chi$. Because $\mathcal{M}_\chi \subset \mathcal{M}_+$,

$$(P_\chi \mu, \mu) \leq (P_+ \mu, \mu), \quad (P \mu, \mu) \leq (P_+ \mu, \mu),$$

where P_+ is the projection operator of $\mathcal{M}_+$. From the last inequality it follows that

$$\mathcal{M} \subset \mathcal{M}_+ , \quad \mathcal{M} \equiv P \mathcal{M} .$$

But $P_\chi P = P_\chi = P P_\chi$, and hence $\mathcal{M}_\chi \subset \mathcal{M}$ which gives

$$\mathcal{M}_+ \subset \mathcal{M} , \quad \mathcal{M}_+ = \mathcal{M} .$$

Consequently $P = P_+$. The statements concerning a non-increasing system can be proved without difficulty when we observe that $\{I - P_\chi\}$ in such case is non-decreasing and the results just obtained are applicable.

7. Geometric significance of functional Hellinger integrals

THEOREM 15. Let ξ_t ($-\infty < t < +\infty$) be a one-parameter family of elements in $\mathcal{M}$ satisfying the conditions that $(\xi_s, \xi_t) = \|\xi_s\|^2$ for $s \leq t$ and that the greatest lower bound of $\|\xi_t\|^2$ is zero. Then for every ξ we have

$$P_{\mathcal{M}} \xi = \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d(\xi_t, \xi_t)} ,$$

where $\mathcal{M}$ is the complete linear extension space of the elements ξ_t , and the integral exists as a strong limit in the sense of finer divisions.

It should be noted that $\alpha(t) \equiv (\xi_t, \xi_t)$ is a monotone increasing function of t , since

$$(\xi_s, \xi_t) = \|\xi_s\|^2, \quad \|\xi_t\|^2 - \|\xi_s\|^2 = \|\xi_t - \xi_s\|^2 > 0$$

for $t \geq s$. According to a previous theorem (Chapter I) the Fourier coefficients (ξ, ξ_t) of ξ form a modular function relative to $\epsilon(st) \equiv (\xi_s, \xi_t)$. Let σ be a finite set of real numbers on $(-\infty, +\infty)$, and t_σ be a real number such that $\alpha(t_\sigma)$ is the minimum of $\alpha(t)$ on σ . We shall use the expression $\frac{a}{s}$, to denote a/s for $s \neq 0$ and zero when $s = 0$. Then by results concerning generalized Hellinger integrals in general analysis,

$$\int_{\sigma} \frac{(\xi, \xi_{t_\sigma})(\xi_{t_\sigma}, \xi)}{\alpha(t_\sigma)} = 0,$$

where σ swells in an infinite set E with $\underline{BE} = -\infty$. As

$$\begin{aligned} \left\| \frac{(\xi, \xi_t)\xi_t}{\alpha(t)} \right\|^2 &= \frac{(\xi, \xi_t)(\xi_t, \xi)}{\alpha(t)} \frac{\|\xi_t\|^2}{\alpha(t)} = \frac{(\xi, \xi_t)(\xi_t, \xi)}{\alpha(t)} \frac{(\xi_t, \xi_t)}{(\xi_t, \xi_t)} \\ &= \frac{(\xi, \xi_t)(\xi_t, \xi)}{\alpha(t)}, \end{aligned}$$

we have clearly

$$\int_{\sigma} \frac{(\xi, \xi_{t_\sigma})\xi_{t_\sigma}}{\alpha(t_\sigma)} = 0.$$

By direct computation we find

$$P_{\mathcal{H}_\sigma} \xi = \frac{(\xi, \xi_{t_\sigma})\xi_{t_\sigma}}{\alpha(t_\sigma)} + H_\sigma \xi$$

where $\mathcal{H}_\sigma$ is the subspace which has as basis the finite set of elements ξ_s with s varying over σ , and

$$\begin{aligned} H_\sigma \xi &\equiv \sum_{m=1}^{n-1} \frac{[(\xi, \xi_{s_{m+1}}) - (\xi, \xi_{s_m})](\xi_{s_{m+1}} - \xi_{s_m})}{\alpha(s_{m+1}) - \alpha(s_m)}, \equiv \sum_{\Delta}^{\sigma} \frac{\Delta(\xi, \xi_t)\Delta\xi_t}{\Delta\alpha(t)}, \\ \sigma &= [s_1 \leq s_2 \leq \dots \leq s_n]. \end{aligned}$$

Now consider a partition π of $(-\infty, +\infty)$ whose points $(-\infty, +\infty$ being excluded) form a denumerably infinite set E_π . Corresponding to these points we have an infinite set of elements ξ_t , t taking values in E_π . This set of elements, ξ_t , gives rise to

a projection operator $P_{E\pi}$. Also, to every finite set τ of $\mathcal{K}$ there is a finite set σ_τ in $(-\infty, +\infty)$ such that the set $[t_\tau]$ with t in σ_τ coincides with τ . As

$$P_{\pi}\{ \} = \left| \frac{S}{\tau} P_{\tau} \right\{ \} = \left| \frac{S}{\sigma_\tau} P_{\sigma_\tau} \right\{ \} = \left| \frac{S}{\sigma} P_{\sigma} \right\{ \},$$

there exists a σ_e for every e such that for every $\sigma \supset \sigma_e$

$$\| P_{\sigma}\{ \} - P_{\pi}\{ \} \| \leq e.$$

Let π_e be a partition whose $E_{\pi_e} \supset \sigma_e$. Now consider a $\pi \supset \pi_e$. Then $E_{\pi} \supset \sigma_e$, and

$$\begin{aligned} P_{E\pi}\{ \} &= \left| \frac{S}{\sigma} P_{\sigma} \right\{ \} = \left| \frac{S}{\sigma \supset \sigma_e} P_{\sigma} \right\{ \}, \quad P_{E\pi}\{ \} - P_{\pi}\{ \} = \left| \frac{S}{\sigma} (P_{\sigma}\{ \} - P_{\pi}\{ \}) \right\{ \}, \\ \| P_{E\pi}\{ \} - P_{\pi}\{ \} \| &= \frac{L}{\sigma} \| P_{\sigma}\{ \} - P_{\pi}\{ \} \|, \end{aligned}$$

where σ swells in the infinite set $\mathcal{K}$. But $\| P_{\sigma}\{ \} - P_{\pi}\{ \} \| \leq e$ for $\sigma \supset \sigma_e$. Consequently

$$\frac{L}{\sigma} \| P_{\sigma}\{ \} - P_{\pi}\{ \} \| \leq e, \quad \| P_{E\pi}\{ \} - P_{\pi}\{ \} \| \leq e,$$

where $E_{\pi} \supset E_{\pi_e}$. Thus for E_{π} swelling in $(-\infty, +\infty)$

$$\left| \frac{S}{E_{\pi}} P_{E_{\pi}} \right\{ \} = P_{\mathcal{K}}\{ \}.$$

Moreover,

$$P_{E\pi}\{ \} = \left| \frac{S}{\sigma} P_{\sigma} \right\{ \} = \left| \frac{S}{\sigma} \frac{(\xi, \xi_{\sigma}) \xi_{\sigma}}{\alpha(\xi_{\sigma})} + \frac{L}{\sigma} H_{\sigma} \right\{ \} = 0 + \frac{E_{\pi}}{\Delta} \frac{\Delta(\xi, \xi_x) \Delta \xi_x}{\Delta \alpha(x)}.$$

So, the limit

$$\left| \frac{S}{E_{\pi}} \frac{E_{\pi}}{\Delta} \frac{\Delta(\xi, \xi_x) \Delta \xi_x}{\Delta \alpha(x)} \right| \equiv \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d \xi_t}{d \alpha(t)}$$

exists and is equal to $P_{\mathcal{K}}\{ \}$.

THEOREM 16. If ξ_t satisfies the conditions mentioned in the preceding theorem and if (ξ_t, ξ_t) is continuous, then the functional Hellinger integral

$$\int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d(\xi_t, \xi_t)}$$

exists as a strong limit in the sense of norm and equals $P_{\pi\xi}$.

First, we show that the integral exists as a weak limit in the sense of norm. Define $\varphi_t \equiv (\xi, \xi_t)$. From the previous theorem

$$\int_{-\infty}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}$$

exists in the sense of finer divisions, $\alpha(t) \equiv (\xi_t, \xi_t)$. Let s be its value. Then there is a π_e of $(-\infty, +\infty)$ such that

$$(1) \quad 0 \leq s - \sum_{\Delta_m} \frac{\pi_e}{\Delta_m} \frac{\Delta_m \varphi_t \Delta_m \bar{\varphi}_t}{\Delta_m \alpha(t)} \leq e.$$

For this infinite series there exists an n_e such that

$$(2) \quad 0 \leq \sum_{\Delta_m} \frac{\pi_e}{\Delta_m} \frac{\Delta_m \varphi_t \Delta_m \bar{\varphi}_t}{\Delta_m \alpha(t)} - \sum_{m=-n_e}^{+n_e} \frac{\Delta_m \varphi_t \Delta_m \bar{\varphi}_t}{\Delta_m \alpha(t)} \leq e.$$

Let the left end point of Δ_{-n_e} be a_e and the right end point of Δ_{n_e} be b_e . By the Cauchy-Lagrange inequality it can be shown (see Hellinger's thesis (1907), pp. 27-8) that

$$\frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq \frac{\Delta}{\delta} \frac{\delta \varphi_t \delta \bar{\varphi}_t}{\delta \alpha(t)}.$$

So for partitions π_0, π_1, π_2 of $[a, b]$, $(-\infty, a]$, $[b, +\infty)$ respectively we have

$$(3) \quad 0 \leq \sum_{\Delta} \frac{\pi_0}{\Delta} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq \int_a^b \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq \int_{-\infty}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)},$$

$$(3) \quad 0 \leq \sum_{\Delta}^{\pi_1} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq \int_{-\infty}^a \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq \int_{-\infty}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}$$

$$0 \leq \sum_{\Delta}^{\pi_2} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq \int_b^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq \int_{-\infty}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}$$

Hence from (1)·(2)·(3)

$$0 \leq \left(\int_{-\infty}^{+\infty} - \int_{a_e}^{b_e} \right) \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq 2e,$$

$$(4) \quad 0 \leq \int_{-\infty}^{a_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq 2e, \quad 0 \leq \int_{b_e}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \leq 2e,$$

because

$$\int_{-\infty}^{+\infty} = \int_{-\infty}^{a_e} + \int_{a_e}^{b_e} + \int_{b_e}^{+\infty}$$

Consider the integral

$$\int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}.$$

For this there is a d_e' such that for any π^* of $[a_e, b_e]$ with $N\pi^* \leq d_e'$ we have

$$(5) \quad 0 \leq \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \sum_{\Delta}^{\pi^*} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq e,$$

since $\int_a^b d\varphi_t d\bar{\varphi}_t / d\alpha(t)$ exists in the sense of norm (as shown in general analysis). On account of continuity of the numerical Hellinger integral with a continuous base,

$$\int_{\Delta_{a_e}} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}, \quad \int_{\Delta_{b_e}} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)}$$

are both less than e when the intervals $\Delta_{a_e}, \Delta_{b_e}$ containing a_e, b_e are smaller than d_e'' . Let π be a partition of $(-\infty, +\infty)$ with $N\pi \leq d_e$ where d_e is the smaller of d_e' and d_e'' . In case π does not have a_e, b_e as dividing points, insert these points into π and get π_{π}^* . Let a_{π}, b_{π} be partition points in π nearest to

a_e, b_e respectively and such that $[a_e, b_e] \subset [a_\pi, b_\pi]$. Let $\Delta_{a_e}, \Delta_{b_e}$ be intervals of π containing a_e, b_e , and $\Delta'_{\pi_e}, \Delta''_{\pi_e}$ be the intervals of π_π^* with a_e, b_e as their respective left and right end points, so that $\Delta'_{\pi_e} \subset \Delta_{a_e}, \Delta''_{\pi_e} \subset \Delta_{b_e}$. Then

$$\begin{aligned} & \left| \sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} - \sum_{\Delta \subset [a_e, b_e]} \frac{\pi_\pi^*}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| = \\ & = \left| \frac{\Delta_{a_e} \varphi_t \Delta_{a_e} \bar{\varphi}_t}{\Delta_{a_e} \alpha(t)} + \frac{\Delta_{b_e} \varphi_t \Delta_{b_e} \bar{\varphi}_t}{\Delta_{b_e} \alpha(t)} - \left[\frac{\Delta'_{\pi_e} \varphi_t \Delta'_{\pi_e} \bar{\varphi}_t}{\Delta'_{\pi_e} \alpha(t)} + \frac{\Delta''_{\pi_e} \varphi_t \Delta''_{\pi_e} \bar{\varphi}_t}{\Delta''_{\pi_e} \alpha(t)} \right] \right| \\ & \leq \left| \frac{\Delta_{a_e} \varphi_t \Delta_{a_e} \bar{\varphi}_t}{\Delta_{a_e} \alpha(t)} - \frac{\Delta'_{\pi_e} \varphi_t \Delta'_{\pi_e} \bar{\varphi}_t}{\Delta'_{\pi_e} \alpha(t)} \right| + \left| \frac{\Delta_{b_e} \varphi_t \Delta_{b_e} \bar{\varphi}_t}{\Delta_{b_e} \alpha(t)} - \frac{\Delta''_{\pi_e} \varphi_t \Delta''_{\pi_e} \bar{\varphi}_t}{\Delta''_{\pi_e} \alpha(t)} \right| \\ & \leq \left| \left(\int_{\Delta_{a_e}} + \int_{\Delta'_{\pi_e}} \right) \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \right| + \left| \left(\int_{\Delta_{b_e}} + \int_{\Delta''_{\pi_e}} \right) \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} \right| \end{aligned}$$

Therefore

$$(6) \quad \left| \sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} - \sum_{\Delta \subset [a_e, b_e]} \frac{\pi_\pi^*}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| \leq |e + e| + |e + e|.$$

Now by (5),

$$0 \leq \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \sum_{\Delta \subset [a_e, b_e]} \frac{\pi_\pi^*}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \leq e.$$

Since

$$\begin{aligned} & \left| \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| \\ & \leq \left| \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \sum_{\Delta \subset [a_e, b_e]} \frac{\pi_\pi^*}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| + \left| \left(\sum_{\Delta \subset [a_e, b_e]} \frac{\pi_\pi^*}{\Delta \alpha(t)} - \sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} \right) \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| \end{aligned}$$

we have from (6)

$$(7) \quad \left| \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} \right| \leq e + 4e.$$

From the simple equality

$$\sum_{\Delta} \frac{\pi}{\Delta \alpha(t)} \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)} = \left(\sum_{\Delta \subset [a_\pi, b_\pi]} \frac{\pi}{\Delta \alpha(t)} + \sum_{\Delta \subset (-\infty, a_\pi]} \frac{\pi}{\Delta \alpha(t)} + \sum_{\Delta \subset [b_\pi, +\infty)} \frac{\pi}{\Delta \alpha(t)} \right) \frac{\Delta \varphi_t \Delta \bar{\varphi}_t}{\Delta \alpha(t)}$$

we derive further

$$\begin{aligned} \left| \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \frac{\pi}{\Delta} \frac{\Delta\varphi_t \Delta\bar{\varphi}_t}{d\alpha(t)} \right| &= \left| \int_{a_e}^{b_e} - \frac{\pi}{\Delta c[a_\pi, b_\pi]} - \frac{\pi}{\Delta c(-\infty, a_\pi]} - \frac{\pi}{\Delta c[b_\pi, +\infty)} \right| \\ &\leq \left| \int_{a_e}^{b_e} - \frac{\pi}{\Delta c[a_\pi, b_\pi]} \right| + \left| \frac{\pi}{\Delta c(-\infty, a_\pi]} \right| + \left| \frac{\pi}{\Delta c[b_\pi, +\infty)} \right| \\ &\leq \left| \int_{a_e}^{b_e} - \frac{\pi}{\Delta c[a_\pi, b_\pi]} \right| + \left| \int_{-\infty}^{a_\pi} \right| + \left| \int_{b_\pi}^{+\infty} \right|, \end{aligned}$$

which, on account of (7) and (4), gives

$$\left| \int_{a_e}^{b_e} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \frac{\pi}{\Delta} \frac{\Delta\varphi_t \Delta\bar{\varphi}_t}{d\alpha(t)} \right| \leq 5e + 2e + 2e.$$

But clearly from this inequality and (4)

$$\left| \int_{-\infty}^{+\infty} \frac{d\varphi_t d\bar{\varphi}_t}{d\alpha(t)} - \frac{\pi}{\Delta} \frac{\Delta\varphi_t \Delta\bar{\varphi}_t}{d\alpha(t)} \right| \leq \left| \int_{-\infty}^{+\infty} - \int_{a_e}^{b_e} \right| + \left| \int_{a_e}^{b_e} - \frac{\pi}{\Delta} \right| \leq 2e + 9e,$$

for every π of $(-\infty, +\infty)$ with $N\pi \leq d_e$. Consequently the numerical Hellinger integral over $(-\infty, +\infty)$ exists in the sense of norm. Polarizing this integral we see that the functional integral

$$\int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)}$$

exists as a weak limit in the sense of norm, since

$$\left\| \frac{\pi}{\Delta} \frac{\Delta(\xi, \xi_t) \Delta\xi_t}{d\alpha(t)} \right\| = \|P_{E\pi} \xi\| \leq \|\xi\|$$

To establish the strong convergence we start with

$$\frac{W}{N\pi=0} P_{E\pi} \xi = \frac{W}{N\pi=0} \frac{\pi}{\Delta} \frac{\Delta(\xi, \xi_t) \Delta\xi_t}{d\alpha(t)} = \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)}.$$

Now

$$\frac{W}{N\pi=0} P_{E\pi} \xi = \frac{W}{E_\pi} P_{E\pi} \xi = \frac{S}{E_\pi} P_{E\pi} \xi = P_\pi \xi$$

so that

$$\begin{aligned} \left(\int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)} \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)} \right) &= (P_{E\pi} \xi, P_{E\pi} \xi) = (P_{E\pi} \xi, \xi) \\ &= \left(\int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)}, \xi \right) = \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d(\xi_t, \xi)}{d\alpha(t)}. \end{aligned}$$

$$\text{As } (P_{E\pi} \xi, P_{E\pi} \xi) = (P_{E\pi} \xi, \xi),$$

$$\begin{aligned} \sum_{N\pi=0}^L (P_{E\pi} \xi, P_{E\pi} \xi) &= \sum_{N\pi=0}^L (P_{E\pi} \xi, \xi) = \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d(\xi_t, \xi)}{d\alpha(t)} \\ &= \left(\int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)}, \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)} \right). \end{aligned}$$

Thus

$$\sum_{N\pi=0}^L \| P_{E\pi} \xi \|^2 = \left\| \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)} \right\|^2,$$

and hence

$$\sum_{N\pi=0}^L P_{E\pi} \xi = \sum_{N\pi=0}^L \frac{\pi}{\Delta} \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \alpha(t)} = \int_{-\infty}^{+\infty} \frac{d(\xi, \xi_t) d\xi_t}{d\alpha(t)}.$$

CHAPTER III

BOUNDED FUNCTIONS OF A BOUNDED HERMITIAN OPERATOR

Besides its intrinsic importance, the Riesz extension of numerical functions to operatorial functions has a natural suggestiveness that is particularly well adapted to the structure study of hermitian operators.

In the present chapter we consider a bounded hermitian operator Ω (in a complete unitary space $\mathcal{M}$) with

$$\overline{M} \equiv \overline{M}\Omega, \quad \underline{M} = \underline{M}\Omega,$$

and $[\underline{M}, \overline{M}]$ as the closed interval of variation. The functions used are all real bounded Baire functions on the variation interval.

1. Polynomials

Associated with a real polynomial

$$P(t) = \sum_{k=1}^m r_k t^k + r_0$$

we define an operatorial polynomial

$$P(\Omega) \equiv \sum_{k=1}^m r_k \Omega^k + r_0 I.$$

Then, any bounded linear operator on $\mathcal{M}$ permutable with Ω is also permutable with $P(\Omega)$. If Ω, Φ are permutable bounded hermitian operators on $\mathcal{M}$, then $P(\Omega), Q(\Phi)$ are also permutable. For polynomials $P(t), Q(t)$ we define

$$L(t) \equiv rP(t) = \sum_k r r_k t^k, \quad M(t) \equiv P(t) + Q(t) = \sum_k (r_k + s_k) t^k,$$

$$N(t) \equiv P(t)Q(t) = \sum_k \left(\sum_{i+j=k} r_i s_j \right) t^k.$$

Hence setting $\Omega^0 \equiv I$ we find

$$\begin{aligned} L(\Omega) &= \sum_k r r_k \Omega^k = r \sum_k r_k \Omega^k = rP(\Omega) \\ M(\Omega) &= \sum_k (r_k + s_k) \Omega^k = \sum_k r_k \Omega^k + \sum_k s_k \Omega^k = P(\Omega) + Q(\Omega) \\ N(\Omega) &= \sum_k \left(\sum_{i+j=k} r_i r_j \right) \Omega^k = \sum_k \left(\sum_{i+j=k} r_i s_j \right) \Omega^i \Omega^j \\ &= \left(\sum_i r_i \Omega^i \right) \left(\sum_j s_j \Omega^j \right) = P(\Omega)Q(\Omega). \end{aligned}$$

THEOREM 1. If $P(t)$ is a non-negative polynomial on the variation interval of Ω then $P(\Omega)$ is a bounded positive hermitian operator.

If $\underline{M} = \overline{M} = s$ then by a previous theorem concerning hermitian operators

$$\Omega = sI, \quad P(\Omega) = P(s) \cdot I,$$

so that the present theorem evidently holds for this case. Now suppose that $\underline{M} \neq \overline{M}$. Let us first consider the case where $P(t) > 0$ on $[\underline{M}, \overline{M}]$. Then $P(u)$ is a properly positive polynomial on $[0, 1]$ if we put

$$u \equiv (t - \underline{M})/(\overline{M} - \underline{M}).$$

According to a theorem of Hausdorff¹ this polynomial can be written in the form

$$P(u) = \sum_{k=0}^n s_k u^k (1-u)^{n-k},$$

where n is an integer not less than m and the coefficients r_k are all positive. It follows that

¹Math. Zeit. 9 (1921), pp. 98-9.

$$P(t) = P(u) = \sum_{k=0}^n s_k \left(\frac{t - \underline{M}}{\overline{M} - \underline{M}} \right)^k \left(\frac{\overline{M} - t}{\overline{M} - \underline{M}} \right)^{n-k} = \sum_{k=0}^n c_k (t - \underline{M})^k (\overline{M} - t)^{n-k},$$

$$c_k \equiv s_k / (\overline{M} - \underline{M})^n.$$

Thus the factors in each term

$$c_k, \quad t - \underline{M}, \quad \overline{M} - t$$

are all positive. Obviously the operators

$$c_k I, \quad \Omega - \underline{M} I, \quad \overline{M} I - \Omega$$

are permutable and positive. Then by a previous theorem on permutable positive operators (Chapter II) we see readily that

$$(\Omega - \underline{M} I)^k, \quad (\overline{M} I - \Omega)^{n-k}, \quad c_k (\Omega - \underline{M} I)^k (\overline{M} I - \Omega)^{n-k}$$

are positive operators. Hence, as a sum of positive operators, $P(\Omega)$ is also positive. Now, suppose that $P(t) \geq 0$ on $[\underline{M}, \overline{M}]$. Clearly, $P(t) + e > 0$ for every $e > 0$, so that $P(\Omega) + eI$ is positive. Consequently $P(\Omega)$ must be positive. Moreover $P(\Omega)$ is bounded since

$$M(P(\Omega)) \leq \sum_{k=1}^m |s_k| (M\Omega)^k + |s_0|.$$

2. Functions of the first order

With this theorem established we are now in a position to treat even the discontinuous operatorial functions. Let $F(t)$ be a bounded function of the first order¹ in the Young sense so that there is a monotone sequence of polynomials $P_n(t)$ converging to $F(t)$. For such a function we define an operatorial function $F(\Omega)$

¹Baire functions of various orders are treated for instance in Hahn, Theorie der reellen Funktionen, Berlin, 1921, pp. 319-392.

as the strong limit of $P_n(\Omega)$ as n becomes infinite. To show that the operatorial limit in question actually exists we have the next theorem.

THEOREM 2. Let $P_n(t)$ be a monotone sequence of polynomials on the variation interval of Ω , converging to a bounded function. Then the strong operatorial limit of $P_n(\Omega)$ exists and is a bounded hermitian operator.

There is no loss of generality if we consider only the case of monotone increasing sequences, for, in the decreasing case - $P_n(t)$ is monotone increasing. Moreover, in the increasing case we may assume that all polynomials are positive, since we can always construct a monotone increasing sequence of positive polynomials $P_n(t)$ from the original increasing sequence $Q_n(t)$ by defining

$$P_n(t) = Q_n(t) - Q_1(t), \quad n = 1, 2, 3, \dots$$

So, we wish to show that $\lim_{n \rightarrow \infty} P_n(\Omega)$ exists and is bounded for a monotone increasing sequence of positive polynomials. As its limit is a bounded function the sequence is uniformly bounded. Let B be a constant upper bound of $P_n(t)$ so that for every m we have

$$0 \leq P_m(t) \leq B, \quad 0 \leq P_{m+1}^2(t) - P_m^2(t), \quad 0 \leq B^2 - P_m^2(t).$$

Hence by Theorem 1,

$$P_{m+1}^2(\Omega) - P_m^2(\Omega), \quad B^2 I - P_m^2(\Omega)$$

are positive operators. Consequently $(P_m^2(\Omega)\mu, \mu)$ is a sequence uniformly bounded and monotone increasing in m . Clearly the limit

$\lim_{m \rightarrow \infty} (P_m^2(\Omega)\mu, \mu)$ exists. Further, for $m \leq n$ we have

$$P_m^2(t) \leq P_m(t)P_n(t) \leq P_n^2(t),$$

and so it follows that

$$(P_m^2(\Omega)\mu, \mu) \leq (P_m(\Omega)P_n(\Omega)\mu, \mu) \leq (P_n^2(\Omega)\mu, \mu).$$

From this we derive the limiting equalities

$$\lim_{m=\infty} (P_m^2(\Omega)\mu, \mu) = \lim_{\substack{m=\infty \\ n=\infty}} (P_m(\Omega)P_n(\Omega)\mu, \mu) = \lim_{n=\infty} (P_n^2(\Omega)\mu, \mu).$$

These imply that $P_n(\Omega)$ converges strongly, because

$$0 = \lim_{\substack{m=\infty \\ n=\infty}} ((P_m(\Omega) - P_n(\Omega))\mu, \mu) = \lim_{\substack{m=\infty \\ n=\infty}} \|P_m(\Omega)\mu - P_n(\Omega)\mu\|^2.$$

Again, because of Theorem 1,

$$(P_m(\Omega)\mu, \mu) \leq B\|\mu\|^2,$$

$$\left(\lim_{n=\infty} P_n(\Omega)\mu, \mu\right) = \lim_{n=\infty} (P_n(\Omega)\mu, \mu) \leq B\|\mu\|^2,$$

and hence $M(\lim_{n=\infty} P_n(\Omega)) \leq B$.

If $F(t)$ is bounded and is the limit of a monotone sequence of polynomials we have established the existence of $F(\Omega)$. However, it still remains a question whether the operatorial function $F(\Omega)$, formed with the aid of a particular monotone sequence of polynomials, is really independent of that sequence. The following theorem provides a suitable answer. Before stating the theorem let us introduce a notation. Suppose that Ω_1, Ω_2 are hermitian operators having linear subspaces $\mathcal{M}_1, \mathcal{M}_2$ as their ranges of definition. Then we write $\Omega_1 \leq \Omega_2$ on $\mathcal{N}$ if $\mathcal{N}$ is the non-empty common intersection of $\mathcal{M}_1, \mathcal{M}_2$ and if $(\Omega_1\xi, \xi) \leq (\Omega_2\xi, \xi)$ for every ξ in $\mathcal{N}$.

THEOREM 3. Let $P_n(t), Q_n(t)$ be two monotone increasing sequences of polynomials converging to the bounded functions $F(t),$

$G(t)$ on the variation interval of Ω . If on that interval $F(t) \leq G(t)$ then $F(\Omega) \leq G(\Omega)$. In particular, $F(\Omega) = G(\Omega)$ when $F(t) = G(t)$ on that interval.

Due to the monotone character of the sequences we have¹

$$P_n(t) - e < \overline{P}_n(t) = F(t) \leq G(t) = \overline{Q}_n(t)$$

for every n , $e > 0$, and t in $[\underline{M}, \overline{M}]$. This shows that there exists an integer k_{net} such that $P_n(t) - e < Q_k(t)$. Therefore, by the continuity of the polynomials, there exists an integer k_{net} and a closed interval Δ_{net} in $[\underline{M}, \overline{M}]$ such that Δ_{net} contains t and

$$P_n(t) - e < Q_k(t).$$

Hence, according to the Heine-Borel theorem, there are some s values of t such that the sum set of the intervals $\Delta_{\text{net}_1} \cdots \Delta_{\text{net}_s}$ covers $[\underline{M}, \overline{M}]$ and

$$P_n(t) - e < Q_{k_{\text{net}_r}}(t), \quad t \text{ in } \Delta_{\text{net}_r}, \quad r = 1, 2, \dots, s.$$

Let k_{ne} be the greatest of $k_{\text{net}_1}, \dots, k_{\text{net}_s}$. Then by the monotonicity of Q_k in $[\underline{M}, \overline{M}]$ we find

$$P_n(t) - e < Q_k(t), \quad t \text{ in } \Delta_{\text{net}_r}, \quad k > k_{\text{ne}}, \quad r = 1, 2, \dots, s.$$

As the sets Δ_{net_r} cover $[\underline{M}, \overline{M}]$,

$$P_n(t) - e < Q_k(t), \quad t \text{ in } [\underline{M}, \overline{M}], \quad k > k_{\text{ne}}.$$

Hence

$$Q_k(t) - P_n(t) + e > 0,$$

which by Theorem 1 gives

¹Except minor changes the following proof is the one originally given in general analysis for a modular hermitian matrix.

$$Q_k(\Omega) - P_n(\Omega) + eI \geq Z, \quad k > k_{ne}.$$

Consequently,

$$\begin{aligned} Q_k(\Omega) &\geq P_n(\Omega) - eI, \\ \lim_{k \rightarrow \infty} Q_k(\Omega) = G(\Omega) &\geq P_n(\Omega) - eI \\ &\geq \lim_{n \rightarrow \infty} P_n(\Omega) - eI = F(\Omega) - eI, \end{aligned}$$

so that

$$G(\Omega) + eI \geq F(\Omega), \quad G(\Omega) \geq F(\Omega).$$

If $F(t)$, $G(t)$ are equal on $[\underline{M}, \overline{M}]$, the equality $F(\Omega) = G(\Omega)$ clearly holds on account of the last inequality.

For convenience of future reference we collect several properties of operatorial functions in a single theorem.

THEOREM 4. The operatorial functions corresponding to the bounded Baire functions of the first order have the following properties:

- (1) $F(\Omega)$ is bounded and hermitian;
- (2) $\underline{B}F \leq \underline{M}F(\Omega) \leq \overline{M}F(\Omega) \leq \overline{B}F$, the bounds of $F(t)$ being taken for t in $[\underline{M}, \overline{M}]$;
- (3) $F(\Omega)$ is permutable with bounded linear operators permutable with Ω ;
- (4) if two hermitian operators Ω_1, Ω_2 are permutable on $\mathcal{M}$ then $F_1(\Omega_1), F_2(\Omega_2)$ are also permutable;
- (5) $(F + G)(\Omega) = F(\Omega) + G(\Omega)$. $F(t), G(t)$ being of the same monotone type;
- (6) $(sF)(\Omega) = sF(\Omega)$;
- (7) $(FG)(\Omega) = F(\Omega)G(\Omega)$, where $F(t), G(t)$ are non-negative and of the same monotone type.

We need only prove properties (2), (3), (7). The rest can be readily obtained from these. For (2) we observe that

$F(\Omega)$ is positive if $F(t) \geq 0$ on $[\underline{M}, \overline{M}]$. By Theorem 3, $F(\Omega) \geq Z$, when $F(t)$ can be expressed as the limit of a monotone increasing sequence. Otherwise consider $-F(t)$ and we reach the same conclusion about $F(\Omega)$. Now $F(t) - \underline{B}F \geq 0$ so that $F(\Omega) - \underline{B}F \cdot I \geq Z$ and $\underline{B}F \cdot I \leq F(\Omega)$. On taking a greatest lower bound we find $\underline{B}F \leq \underline{M}F(\Omega)$. Similar consideration applies to $\overline{B}F$ and $\overline{M}F(\Omega)$. Suppose that Φ is permutable with Ω . Then it is permutable with $P_n(\Omega)$ so that

$$\Phi F(\Omega) = \Phi \left(\lim_{n \rightarrow \infty} P_n(\Omega) \right) = \lim_{n \rightarrow \infty} \Phi P_n(\Omega) = \left(\lim_{n \rightarrow \infty} P_n(\Omega) \right) \Phi = F(\Omega) \Phi,$$

and we have (3). To establish (7), introduce the polynomials

$$\begin{aligned} R_n(t) &\equiv P_n(t) + A, & S_n(t) &\equiv Q_n(t) + B, \\ A &\equiv |\underline{B}P_n(t)|, & B &\equiv |\underline{B}Q_n(t)|. \end{aligned}$$

Then $R_n(t)$, $S_n(t)$ are non-negative polynomials and they form monotone increasing sequences converging to $F(t) + A$, $G(t) + B$ respectively. From (5) we see that

$$\lim_{n \rightarrow \infty} R_n(\Omega) = (F + A)(\Omega) = F(\Omega) + AI,$$

$$\lim_{n \rightarrow \infty} S_n(\Omega) = (G + B)(\Omega) = G(\Omega) + BI.$$

Now $R_n(t)S_n(t)$ is a monotone increasing sequence of non-negative polynomials converging to $(F(t) + A)(G(t) + B)$, since

$$R_n(t)S_n(t) \leq R_{n+1}(t)S_n(t) \leq R_{n+1}(t)S_{n+1}(t).$$

By Theorem 3 it follows that

$$\lim_{n \rightarrow \infty} (R_n S_n)(\Omega) = ((F + A)(G + B))(\Omega).$$

But

$$(R_n S_n)(\Omega) = R_n(\Omega) S_n(\Omega).$$

Hence we have

$$\lim_{n \rightarrow \infty} R_n(\Omega) \lim_{n \rightarrow \infty} S_n(\Omega) = \lim_{n \rightarrow \infty} (R_n(\Omega) S_n(\Omega)) = \lim_{n \rightarrow \infty} (R_n S_n)(\Omega).$$

This gives

$$\begin{aligned} F(\Omega)G(\Omega) + AG(\Omega) + BF(\Omega) + ABI &= (F(\Omega) + AI)(G(\Omega) + BI) \\ &= (F + A)(\Omega)(G + B)(\Omega) = ((F + A)(G + B))(\Omega). \end{aligned}$$

However, on account of (5)•(6),

$$\begin{aligned} ((F + A)(G + B))(\Omega) &= (FG + AG + BF + AB)(\Omega) = (FG)(\Omega) + \\ &\quad AG(\Omega) + BF(\Omega) + ABI, \end{aligned}$$

so that

$$F(\Omega)G(\Omega) = (FG)(\Omega).$$

The operatorial function corresponding to a continuous function has an interesting property which reflects the numerical continuity of the generating function.

THEOREM 5. Suppose that $F(t)$ is a continuous function on a closed interval. Then $F(\Omega)$ is modularly continuous in its operatorial argument, i.e., $M(F(\Omega_1) - F(\Omega_2))$ can be made arbitrarily small when $M(\Omega_1 - \Omega_2)$ becomes sufficiently small, Ω_1, Ω_2 being bounded hermitian operators (not necessarily permutable) with their variation intervals in the continuity interval of $F(t)$.

Let $[a, b]$ be the interval on which $F(t)$ is continuous, and c be the greater of $|a|, |b|$. First consider the integral power of t, t^k . We show that

$$M(\Omega_1^k - \Omega_2^k) \leq M(\Omega_1 - \Omega_2) \cdot k \cdot c^{k-1}.$$

This is certainly obvious for $k = 0$ or 1 . For $k \geq 2$ we have

$$\Omega_1^k - \Omega_2^k = \sum_{i,j=0}^{i+j=k-1} \Omega_1^i (\Omega_1 - \Omega_2) \Omega_2^j,$$

since

$$\sum_{i,j=0}^{\infty} \Omega_1^i (\Omega_1 - \Omega_2) \Omega_2^j = \sum_{i,j=0}^{\infty} \Omega_1^{i+1} \Omega_2^j - \sum_{i,j=0}^{\infty} \Omega_1^i \Omega_2^{j+1}.$$

It follows at once that

$$\begin{aligned} M(\Omega_1^k - \Omega_2^k) &\leq \sum_{i,j=0}^{i+j=k-1} (M\Omega_1)^i M(\Omega_1 - \Omega_2) (M\Omega_2)^j \\ &\leq M(\Omega_1 - \Omega_2) \cdot k \cdot (\text{gr}(M\Omega_1, M\Omega_2))^{k-1} \\ &\leq M(\Omega_1 - \Omega_2) \cdot k \cdot c^{k-1}. \end{aligned}$$

With the aid of the last inequality we derive moreover

$$\begin{aligned} P(t) &\equiv \sum_{k=0}^n s_k t^k, \\ M(P(\Omega_1) - P(\Omega_2)) &= M\left(\sum_{k=1}^n s_k (\Omega_1^k - \Omega_2^k)\right) \leq \sum_{k=1}^n |s_k| M(\Omega_1^k - \Omega_2^k) \\ &\leq \sum_{k=1}^n |s_k| M(\Omega_1 - \Omega_2) \cdot k \cdot c^{k-1} \\ &\leq M(\Omega_1 - \Omega_2) \sum_{k=1}^n k |s_k| c^{k-1}. \end{aligned}$$

Thus $M(P(\Omega_1) - P(\Omega_2)) \leq e$ for $M(\Omega_1 - \Omega_2) \leq d_{eP}$,

$$d_{eP} \equiv e / \left(1 + \sum_{k=1}^n k |s_k| c^{k-1}\right).$$

Now we come to the continuous function $F(t)$. According to a theorem of Weierstrass there exists a monotone increasing sequence of polynomials $P_n(t)$ such that $P_n(t)$ converge uniformly to $F(t)$. Because of this uniform convergence, to every positive e there corresponds an n_e such that

$$\begin{aligned} |P_{n_e}(t) - F(t)| &\leq e, \quad a \leq t \leq b, \\ M(P_{n_e}(\Omega_1) - F(\Omega_1)) &\leq e, \quad M(P_{n_e}(\Omega_2) - F(\Omega_2)) \leq e. \end{aligned}$$

But, as we have just shown, there is a $d_{eP_{n_e}} \equiv d_e$ such that

$$M(P_{n_e}(\Omega_1) - P_{n_e}(\Omega_2)) \leq e, \quad M(\Omega_1 - \Omega_2) \leq d_e.$$

This gives the desired result, for

$$\begin{aligned} M(F(\Omega_1) - F(\Omega_2)) &\leq M(F(\Omega_1) - P_{n_e}(\Omega_1)) + M(P_{n_e}(\Omega_1) - P_{n_e}(\Omega_2)) \\ &\quad + M(P_{n_e}(\Omega_2) - F(\Omega_2)) \leq e + e + e, \end{aligned}$$

whenever $M(\Omega_1 - \Omega_2) \leq d_e$.

As we know, functions of the first order do not form a linear set. Let us form finite linear combinations and thus we obtain an extended class of functions of the first order. It can be readily seen that a function $F(t)$ in this extended class is expressible as the difference of two functions of the first order, $G(t) - H(t)$, where $G(t)$, $H(t)$ are limits of monotone increasing sequences. For $F(t)$ we then define

$$F(\Omega) \equiv G(\Omega) - H(\Omega).$$

It should be noted that $F(\Omega)$ is independent of the decomposition of $F(t)$. Suppose we have two decompositions:

$$G_1(t) - H_1(t) = F(t) = G_2(t) - H_2(t).$$

From this we find

$$G_1(t) + H_2(t) = G_2(t) + H_1(t), \quad G_1(\Omega) + H_2(\Omega) = G_2(\Omega) + H_1(\Omega),$$

as the terms in each sum are first order functions of the same type. The last equality evidently implies

$$G_1(\Omega) - H_1(\Omega) = G_2(\Omega) - H_2(\Omega).$$

THEOREM 6. The operatorial correspondents of functions in the linearly extended class of first order functions have in addition to the properties (1), (2), (3), (4), (6) mentioned in Theorem 4 the following two properties:

$$(5) (F + G)(\Omega) = F(\Omega) + G(\Omega);$$

$$(7) (FG)(\Omega) = F(\Omega)G(\Omega).$$

Properties (1), (3), (4), (5), (6) can be easily seen.

We proceed to establish the others. Clearly

$$\underline{BF} \leq G(t) - H(t) \leq \overline{BF}, \quad \underline{BF} + H(t) \leq G(t) \leq \overline{BF} + H(t).$$

By virtue of Theorem 4 and the last inequalities we have

$$H(\Omega) + \underline{BFI} \leq G(\Omega) \leq H(\Omega) + \overline{BFI}.$$

These imply that

$$\underline{BFI} \leq G(\Omega) - H(\Omega) \leq \overline{BFI},$$

$$\underline{BF}(\mu, \mu) \leq (F(\Omega)\mu, \mu) \leq \overline{BF}(\mu, \mu).$$

So, (2) holds. For (7) we note that if $F(t)$, $G(t)$ are in the extended class, $F(t)G(t)$ belongs to it also, because

$$F(t) = F_1(t) - F_2(t) = F_1'(t) - F_2'(t),$$

$$G(t) = G_1(t) - G_2(t) = G_1'(t) - G_2'(t),$$

$$F(t)G(t) = (F_1'(t)G_1'(t) + F_2'(t)G_2'(t)) - (F_2'(t)G_1'(t) + F_1'(t)G_2'(t)),$$

$$F_1'(t) \equiv F_1(t) + |\underline{BF}_1| + |\underline{BF}_2|, \quad F_2'(t) \equiv F_2(t) + |\underline{BF}_1| + |\underline{BF}_2|,$$

etc. As the first order functions $F_1'(t)$, $F_2'(t)$, $G_1'(t)$, $G_2'(t)$ are non-negative, the last equation for $F(t)G(t)$ gives

$$\begin{aligned} (FG)(\Omega) &= (F_1'G_1')(\Omega) + (F_2'G_2')(\Omega) - (F_2'G_1')(\Omega) - (F_1'G_2')(\Omega) \\ &= F_1'(\Omega)G_1'(\Omega) + F_2'(\Omega)G_2'(\Omega) - F_2'(\Omega)G_1'(\Omega) - F_1'(\Omega)G_2'(\Omega) \\ &= F(\Omega)G(\Omega). \end{aligned}$$

3. Separation operators and integral representation

Consider the function $E_t(r)$ which takes the value 1 for r less than t and vanishes elsewhere. This lower semi-continuous

function is non-negative and naturally of the first order. Form its operatorial correspondent $E_t(\Omega) \equiv E_t$. Then E_t is positive hermitian and

$$E_t^2 = (E_t E_t)(\Omega) = E_t(\Omega) = E_t,$$

since $E_t^2(r) = E_t(r)$, so that E_t is a projection operator. Further this operator makes a separation of Ω as we show in the next theorem.

THEOREM 7. To every bounded hermitian operator Ω there corresponds another E_t having the following properties:

- (1) E_t is a projection operator;
- (2) it is permutable with any bounded linear operator permutable with Ω ;
- (3) $(\Omega - tI)E_t$ and $(\Omega - tI)(I - E_t)$ are respectively negative and positive;
- (4) we have $E_t \mu = 0$ whenever $\Omega \mu = t \mu$.

Properties (1) and (2) are now obvious due to previous considerations. In (3) we observe that $(\Omega - tI)E_t$ and $(\Omega - tI)(I - E_t)$ correspond to the numerical functions $(r - t)E_t(r)$ and $(r - t)(1 - E_t(r))$. Now these functions are negative and positive respectively and so are their operatorial correspondents, by Theorem 6. For (4) we start with the simple fact that

$$P(\Omega)\mu = P(t)\mu, \quad \Omega\mu = t\mu.$$

Passing to the limit we find

$$F(\Omega)\mu = F(t)\mu$$

when $\Omega\mu = t\mu$. Thus (4) is readily obtained if we recall that $E_t(r) = 0$ for $r = t$.

Remark. In Theorem 7 the system of operators E_t may be called a left system. Now let $E_{*t}(r)$ be 1 for r not greater than t and 0 otherwise. If we form the operatorial correspondent $E_{*t}(\Omega)$, then $E_{*t} \equiv E_{*t}(\Omega)$ satisfies conditions (1)-(3) of that theorem and instead of (4) the new operator has the property

$$(4') \quad \Omega\mu = t\mu \text{ implies } E_{*t}\mu = \mu.$$

These operators constitute a right system. The treatments of left and right systems are however entirely parallel, and, in what follows we consider the left system only.

For further study we need the integral representation of operators and the next theorem gives a preliminary result of that type.

THEOREM 8. For a function $F(t)$ in the extended class of first order functions the following equality holds:

$$(F(\Omega)\mu, \xi) = \int_{\underline{M}}^{\overline{M}} F(t) d(E_t\mu, \xi),$$

where the integral is a Lebesgue-Stieltjes integral taken over the closed interval of variation.

Consider the variation of $(E_t\mu, \mu)$ over an arbitrary subinterval Δ in $[\underline{M}, \overline{M}]$. Since

$$\Delta(E_{\chi}\mu, \mu) = (\Delta E_{\chi}\mu, \mu), \quad \Delta E_{\chi} = (\Delta E_{\chi})(\Omega),$$

and $\Delta E_t(r)$ is positive, $\Delta(E_t\mu, \mu)$ is also positive. Thus $(E_t\mu, \mu)$ is monotone increasing and hence $(E_t\mu, \xi)$ is a function of limited variation by polarization.

To establish the equality in the theorem let us first show that it holds for a Stieltjes integral with a continuous integrand function. For a value t_{Δ} in Δ and a partition π of $[\underline{M}, \overline{M} + d]$ we have from Theorem 6 when $t > \overline{M}$

$$F(t_{\Delta}) \Delta E_t = (F(t_{\Delta}) \Delta E_t)(\Omega), \quad F(t) \equiv F(\overline{M}),$$

$$\begin{aligned} \lim_{N\pi=0} \left(\left(\sum_{\Delta}^{\pi} F(t_{\Delta}) \Delta E_t \right)(\Omega) \mu, \xi \right) &= \lim_{N\pi=0} \sum_{\Delta}^{\pi} F(t_{\Delta}) \cdot \Delta (E_t \mu, \xi) \\ &= S \int_{\underline{M}}^{\overline{M}+d} F(t) d(E_t \mu, \xi), \end{aligned}$$

where the integral always exists as $F(t)$ is continuous and $(E_t \mu, \xi)$ is a function of limited variation. Put

$$F_{\pi}(r) \equiv \sum_{\Delta}^{\pi} F(t_{\Delta}) \Delta E_t(r).$$

Then $F_{\pi}(r)$, being a finite sum of first order functions, certainly belongs to the linearly extended class. Moreover, for every r in the variation interval of Ω there exists uniquely a subinterval Δ_r such that r lies in Δ_r and $F_{\pi}(r) = F(t_{\Delta_r})$. So, by the continuity of $F(t)$ we find

$$|F(r) - F_{\pi}(r)| = |F(r) - F(t_{\Delta_r})| \leq \epsilon \text{ when } |r - t_{\Delta_r}| \leq N\pi \leq d_{\epsilon}.$$

By virtue of Theorem 6 we deduce that

$$\begin{aligned} |F(\Omega) \mu, \mu) - (F_{\pi}(\Omega) \mu, \mu)| &\leq \epsilon, \\ (F(\Omega) \mu, \mu) &= \lim_{N\pi=0} (F_{\pi}(\Omega) \mu, \mu) = \lim_{N\pi=0} \left(\left(\sum_{\Delta}^{\pi} F(t_{\Delta}) \Delta E_t \right)(\Omega) \mu, \mu \right) \\ &= S \int_{\underline{M}}^{\overline{M}+d} F(t) d(E_t \mu, \mu), \end{aligned}$$

which yields the desired equality after polarization.

But the Stieltjes integral is equal to the Lebesgue-Stieltjes integral.¹ From their definitions

$$E_t(r) = 1, \quad c > \overline{M}, \quad \underline{M} \leq r \leq \overline{M}.$$

It follows that

¹See Lebesgue, Lecons sur l'integration, Paris, 1928, pp. 253-90.

$$E_t(\Omega) = E_t = I, \quad t > \overline{M},$$

which has the consequence that the measure of $(\overline{M}, \overline{M} + d]$ with respect to $(E_t\mu, \xi)$ is always zero. So, we can write

$$(F(\Omega)\mu, \xi) = \int_{\underline{M}}^{\overline{M}} F(t) d(E_t\mu, \xi),$$

the integral being taken over a closed interval. Now consider a function $F(t)$ of the first order. As there is a uniformly bounded monotone sequence of polynomials $P_n(t)$ converging to $F(t)$,

$$\begin{aligned} (F(\Omega)\mu, \xi) &= \lim_{n \rightarrow \infty} (P_n(\Omega)\mu, \xi) = \lim_{n \rightarrow \infty} \int_{\underline{M}}^{\overline{M}} P_n(t) d(E_t\mu, \xi) \\ &= \int_{\underline{M}}^{\overline{M}} F(t) d(E_t\mu, \xi), \end{aligned}$$

by a theorem in Lebesgue-Stieltjes integration. Obviously this equality applies to any function in the extended class.

4. Baire functions of higher classes

The operatorial functions corresponding to continuous functions have the properties listed in Theorem 6. Now with the aid of integral representation we can extend these results to all bounded Baire functions of higher classes. What is a Baire operatorial function? We define it by induction.

Let $F(t)$ be a bounded function of the α -th class with $F_n(t)$ as an approximating sequence of functions uniformly bounded of lower classes. Then the strong limit of $F_n(\Omega)$ is called the operatorial correspondent of $F(t)$ and denoted by $F(\Omega)$.

THEOREM 9. The operatorial correspondent of any bounded Baire function exists and satisfies the seven properties stated in Theorem 6 and the equality in Theorem 8.

Let B be the constant least upper bound of the functions $|F_n(t)|$, and, to carry out an induction proof, suppose that all bounded operatorial functions of lower classes satisfy the properties and the equality mentioned in this theorem. Then by the properties in Theorem 6,

$$\|F_n(\Omega)\mu\|^2 = (F_n(\Omega)\mu, F_n(\Omega)\mu) = (F_n^2(\Omega)\mu, \mu) \leq B^2 \cdot (\mu, \mu).$$

Further, due to the convergence of the bounded sequence,

$$\lim_{n \rightarrow \infty} (F_n(\Omega)\mu, \xi) = \lim_{n \rightarrow \infty} \int_{\underline{M}}^{\overline{M}} F_n(t) d(E_t \mu, \xi) = \int_{\underline{M}}^{\overline{M}} F(t) d(E_t \mu, \xi).$$

Combining these two results we see that $F_n(\Omega)\mu$ converge weakly. On the other hand, the functions $(F_m(t) - F_n(t))^2$ are of lower classes and uniformly bounded with

$$\lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} (F_m(t) - F_n(t))^2 = 0.$$

Applying the weak convergence result to $(F_m - F_n)^2(\Omega)$ we find

$$0 = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} ((F_m - F_n)^2(\Omega)\mu, \mu) = \lim_{\substack{m \rightarrow \infty \\ n \rightarrow \infty}} \|(F_m(\Omega) - F_n(\Omega))\mu\|^2.$$

Thus the operatorial functions $F_n(\Omega)$ converge strongly. This establishes the existence of $F(\Omega)$. The equality in Theorem 8 becomes obvious when we observe that

$$\begin{aligned} (F(\Omega)\mu, \xi) &= \left(\lim_{n \rightarrow \infty} F_n(\Omega)\mu, \xi \right) = \lim_{n \rightarrow \infty} (F_n(\Omega)\mu, \xi) \\ &= \int_{\underline{M}}^{\overline{M}} F(t) d(E_t \mu, \xi). \end{aligned}$$

Properties (3), (4) in Theorem 6 can be shown without difficulty because of the sequence definition of $F(\Omega)$. That $F(\Omega)$ is bounded is also clear, since

$$|(F(\Omega)\mu, \mu)| = \left| \lim_{n \rightarrow \infty} (F_n(\Omega)\mu, \mu) \right| \leq B \cdot (\mu, \mu).$$

Now by the mean-value theorem for Lebesgue-Stieltjes integrals

$$\begin{aligned} \underline{BF} \cdot (\mu, \mu) &\leq \int_{\underline{M}}^{\overline{M}} F(t) d(E_t \mu, \mu) = (F(\Omega)\mu, \mu) \leq \overline{BF} \cdot (\mu, \mu), \\ \int_{\underline{M}}^{\overline{M}} 1 d(E_t \mu, \mu) &= (\mu, \mu), \end{aligned}$$

which gives property (2). For the other properties we note that

$$\begin{aligned} (F + G)(\Omega) &= \lim_{n \rightarrow \infty} (F_n + G_n)(\Omega) = \lim_{n \rightarrow \infty} (F_n(\Omega) + G_n(\Omega)) \\ &= F(\Omega) + G(\Omega), \end{aligned}$$

$$(sF)(\Omega) = \lim_{n \rightarrow \infty} (sF_n)(\Omega) = \lim_{n \rightarrow \infty} sF_n(\Omega) = sF(\Omega),$$

$$(FG)(\Omega) = \lim_{n \rightarrow \infty} (F_n G_n)(\Omega) = \lim_{n \rightarrow \infty} F_n(\Omega) G_n(\Omega) = F(\Omega) G(\Omega),$$

because, two bounded sequences of operatorial functions converge to the same limit whenever the corresponding sequences of numerical functions converge to the same function, as seen from the sequences of integrals.

5. Accordant functions of an operator

In the foregoing sections our definition of an operatorial function is such that functions of an $n \times n$ singular matrix may be non-singular. Such a definition has met some objections.¹ In view of these we consider another definition in this section, and to distinguish the new functions from the previous ones we call these accordant functions. However, the two types of functions have definite relations as will be seen presently.

To make the desired definition we need an auxiliary operator associated with Ω . It is the projection operator of Ω ,

¹C. E. Cullis, Matrices and Determinoids, Vol. III, Part I, 1925, pp. 632-4.

$$Q \equiv P_{\mathcal{M}}, \quad \mathcal{N} \equiv \Omega \mathcal{M}$$

For every ξ in $\mathcal{M}$, $(Q\Omega\xi, \xi) = (\Omega\xi, \xi)$ so that

$$Q\Omega = \Omega.$$

On the other hand,

$$\Omega Q = (Q\Omega)^* = \Omega^* = \Omega.$$

Further, if $Q\xi = 0$ then $\Omega\xi = 0$, for,

$$\Omega\xi = \Omega Q\xi = 0.$$

Conversely, $\Omega\xi = 0$ implies $Q\xi = 0$. When $\Omega\xi = 0$ we have also

$$(\mathcal{M}, \Omega\xi) = 0, \quad (\Omega\mathcal{M}, \xi) = 0.$$

Thus ξ is orthogonal to $\Omega\mathcal{M} = \mathcal{N}$, and hence $P_{\mathcal{N}}\xi = 0$. These results we state in the next theorem.

THEOREM 10. Between Ω and Q the following relations hold:

$$(1) \quad Q\Omega = \Omega = \Omega Q.$$

$$(2) \quad \Omega\xi = 0 \text{ if and only if } Q\xi = 0.$$

On account of relation (1) in this theorem, Ω is on $Q\mathcal{M}$ to $Q\mathcal{M}$ so that it operates in the complete unitary space $Q\mathcal{M}$.

Now for

$$P(t) \equiv \sum_{k=1}^m r_k t^k + r_0$$

let us define the accordant polynomial

$$P^*(\Omega) \equiv \sum_{k=1}^m r_k \Omega^k + r_0 Q.$$

Here, of course, Q is the identity operator in $Q\mathcal{M}$. Thus in the present situation we are working entirely with $Q\mathcal{M}$, and many of

the previous results concerning Ω on $\mathcal{M}$ can be at once applied to Ω on $Q\mathcal{M}$. So, we have

THEOREM 11. To every bounded Baire function $F(t)$ there corresponds an accordant operatorial function $F^*(\Omega)$. These correspondents have the six properties stated in Theorem 6. Moreover

$$QF^*(\Omega) = F^*(\Omega) = F^*(\Omega)Q, \quad \underline{MF}(\Omega) \leq \underline{MF}^*(\Omega) \leq \overline{MF}^*(\Omega) \leq \overline{MF}(\Omega).$$

The next theorem establishes a definite relation between the two types of operatorial functions.

THEOREM 12. For $F(\Omega)$ and $F^*(\Omega)$ we have the equation

$$F(\Omega) = F^*(\Omega) + F(0)(I - Q).$$

In proof we first observe that

$$(I - Q) = Z = (I - Q)\Omega, \quad P(\Omega)(I - Q) = P(0)(I - Q), \\ P(\Omega)Q = P^*(\Omega).$$

By passing to the limit and by induction we find

$$F(\Omega)(I - Q) = F(0)(I - Q), \quad F(\Omega)Q = F^*(\Omega).$$

Now

$$QF^*(\Omega) = F^*(\Omega) = F^*(\Omega)Q,$$

so that

$$F^*(\Omega)(I - Q) = Z.$$

These lead to

$$F(\Omega) - F^*(\Omega) = (F(\Omega) - F^*(\Omega))(Q + (I - Q)) \\ = F^*(\Omega) - F^*(\Omega) + F(0)(I - Q) - Z,$$

which gives the desired equation.

CHAPTER IV

EXISTENCE, UNICITY, AND PROPERTIES OF THE SOLUTION

1. Self-adjoint operator

In the present and the ensuing chapters we shall consider certain unbounded hermitian operators to be characterized shortly, the space being again complete unitary over a number system $\mathcal{N}$ (real, complex, or quaternionic).

First of all, we should remark that a space $\mathcal{N}$ over the reals can be easily extended to another $\mathcal{N}$ over the complexes. This $\mathcal{N}$ consists of all ordered pairs $\{\xi, \zeta\}$ from $\mathcal{N}$ such that

$$\begin{aligned} c &\equiv a + ib, & c\mu &\equiv \{a\xi - b\zeta, b\xi + a\zeta\}, \\ \{\xi_1, \zeta_1\} + \{\xi_2, \zeta_2\} &\equiv \{\xi_1 + \xi_2, \zeta_1 + \zeta_2\}, \\ [\{\xi_1, \zeta_1\}, \{\xi_2, \zeta_2\}] &\equiv (\xi_1, \zeta_2) + (\zeta_1, \xi_2) + i(\zeta_1, \zeta_2) - i(\xi_1, \xi_2). \end{aligned}$$

Then $\mathcal{N}$ satisfies all the defining properties of a complete unitary space. A linear operator $\mathfrak{F}$ in $\mathcal{N}$ can be similarly extended to $\mathcal{N}$ by the equation

$$\mathfrak{F}\{\xi, \zeta\} \equiv \{\mathfrak{F}\xi, \mathfrak{F}\zeta\}.$$

Because of these facts the next definition applies to the reals also.

We call an operator Ω on $\mathcal{N}(\Omega)$ to self-adjoint when and only when (1) $\mathcal{N}(\Omega)$ is linear, dense on $\mathcal{N}$, (2) Ω is hermitian, and (3) there exists a non-real number w such that $\Omega - wI$ and $\Omega - \bar{w}I$ have reciprocals on the entire space $\mathcal{N}$. Obviously, the reciprocals are linear with respect to the reals.

Further, they are bounded, for

$$(1) \quad \begin{aligned} \Omega \zeta - w \zeta &= \mu, \quad (\Omega - wI)^{-1} \mu = \zeta, \quad w \equiv s + v, \quad v \neq 0, \\ \|\mu\|^2 &= \|\Omega \zeta - w \zeta\|^2 = ((\Omega - sI)\zeta, (\Omega - sI)\zeta) + (-v\zeta, -v\zeta) \\ &\geq v(\zeta, \zeta)\bar{v}, \end{aligned}$$

and hence

$$(2) \quad \|\zeta\|^2 \leq |1/v|^2 \|\mu\|^2.$$

It should be noted that in a commutative number system Ω is self-adjoint if and only if the equality $(\zeta, \Omega \zeta) = (\zeta^*, \zeta)$ for every ζ in $\mathcal{M}(\Omega)$ implies that ζ lies in $\mathcal{M}(\Omega)$ and $\Omega \zeta = \zeta^*$, provided $\mathcal{M}(\Omega)$ is linear dense on $\mathcal{M}$. If Ω is self-adjoint and if $(\zeta, \Omega \zeta) = (\zeta^*, \zeta)$ for every ζ of $\mathcal{M}(\Omega)$, then

$$(\zeta, (\Omega - wI)\zeta) = (\zeta^*, \zeta) - (\zeta, \zeta)\bar{w} = (\zeta^* - \bar{w}\zeta, \zeta).$$

Due to the self-adjointness of Ω the reciprocals $(\Omega - wI)^{-1}$, $(\Omega - \bar{w}I)^{-1}$ are on $\mathcal{M}$ to $\mathcal{M}(\Omega)$. Put $\xi \equiv (\Omega - wI)^{-1}\mu$ in the above equation and obtain

$$(\zeta, \mu) = (\zeta^* - \bar{w}\zeta, (\Omega - wI)^{-1}\mu) = ((\Omega - \bar{w}I)^{-1}(\zeta^* - \bar{w}\zeta), \mu).$$

As this holds for every μ in $\mathcal{M}$ we have

$$\xi = (\Omega - \bar{w}I)^{-1}(\zeta^* - \bar{w}\zeta).$$

But $(\Omega - \bar{w}I)^{-1}$ is on $\mathcal{M}$ to $\mathcal{M}(\Omega)$ so that ξ belongs to $\mathcal{M}(\Omega)$. Moreover, Ω is hermitian and consequently from the second hypothesis

$$(\zeta^*, \xi) = (\zeta, \Omega \xi) = (\Omega \zeta, \xi),$$

which shows that $\zeta^* = \Omega \zeta$.

Now suppose that $\mathcal{M}(\Omega)$ is linear dense on $\mathcal{M}$ and that $(\xi, \Omega \mathcal{M}(\Omega)) = (\xi^*, \mathcal{M}(\Omega))$ implies $\Omega \xi = \xi^*$. By definition of Ω^* , ξ^* belongs to $\mathcal{M}(\Omega^*)$ when $(\xi, \Omega \mathcal{M}(\Omega)) = (\xi^*, \mathcal{M}(\Omega))$. Thus the second part of our assumption on Ω becomes $\Omega = \Omega^*$. Clearly Ω is hermitian, for

$$(\xi_1, \Omega \xi_2) = (\Omega^* \xi_1, \xi_2) = (\Omega \xi_1, \xi_2),$$

where ξ_1, ξ_2 lie in $\mathcal{M}(\Omega) = \mathcal{M}(\Omega^*)$. We note in passing that if these assumptions are verified for the case of the reals they also hold for its corresponding extension to the complexes. So we proceed to work with complex numbers. To show that $\Omega - wI$ has a bounded reciprocal for every non-real w , we observe first that $(\Omega - rI)^* = \Omega - rI$, r being an arbitrary real number, hence $\Omega - wI = (\Omega - rI) - isI$ so that it is of the form $\Omega_1 - isI$. Without loss of generality we need only prove¹ that $\Omega - isI$ ($s \neq 0$) has a bounded reciprocal, when $\Omega^* = \Omega$. To that end we consider the equation

$$\Omega \xi - is\xi = \mu$$

for the unknown ξ with μ given in advance. If $\mu = 0$ then $\xi = 0$ is the only solution. For, let ξ_1 be any solution and hence

$$(\Omega \xi_1, \xi_1) = is(\xi_1, \xi_1),$$

which, due to the reality of $(\Omega \xi_1, \xi_1)$ and (ξ_1, ξ_1) , shows that

$$\|\xi_1\|^2 = (\xi_1, \xi_1) = 0, \quad \xi_1 = 0.$$

¹Here we use an argument of F. Riesz given for the Hilbert space. It is applicable to the present situation, since we have established the orthogonal decomposition theorem.

This fact also means that the solution for ξ is always unique if it exists. Put

$$\pi \equiv (\Omega - i\mathbf{s}\mathbf{I})\mathcal{M}(\Omega).$$

Then the equation has a solution when μ lies in the linear subspace π . We maintain that π is also dense on $\mathcal{M}$. Otherwise, there would be, by the orthogonal decomposition theorem, a ξ in $\mathcal{M}$ such that ξ , different from zero, is orthogonal to π . In consequence we would have

$$(\xi, (\Omega - i\mathbf{s}\mathbf{I})\xi) = 0, \quad (\xi, \Omega\xi) = -(\xi, \xi)i\mathbf{s} = (-i\mathbf{s}\xi, \xi)$$

for every ξ in $\mathcal{M}(\Omega)$. As $\Omega^* = \Omega$, this gives $\Omega\xi = -i\mathbf{s}\xi$ and

$$(\Omega\xi, \xi) = (-i\mathbf{s}\xi, \xi) = -i\mathbf{s}(\xi, \xi),$$

which contradicts the reality of $(\Omega\xi, \xi)$. Now that π is known to be dense we go on to show $\pi = \mathcal{M}$. Let μ be a vector in $\mathcal{M}$ so that there is a sequence $\{\mu_n\}$ from π with

$$(3) \quad \lim_{n \rightarrow \infty} \mu_n = \mu.$$

Associated with this sequence is another $\{\xi_n\}$ such that

$$\Omega\xi_n - i\mathbf{s}\xi_n = \mu_n.$$

We know by (2) that

$$\|\xi_m - \xi_n\| \leq |1/\mathbf{s}| \cdot \|\mu_m - \mu_n\|,$$

and ξ_m converge strongly to a limit ξ by (3). On the other hand, for every ξ in $\mathcal{M}(\Omega)$

$$(\Omega\xi, \xi) = \lim_{n \rightarrow \infty} (\Omega\xi, \xi_n) = \lim_{n \rightarrow \infty} (\xi, \Omega\xi_n) = \lim_{n \rightarrow \infty} (\xi, \mu_n + i\mathbf{s}\xi_n)$$

$$= (\xi, \mu + i s \xi),$$

from which we infer that ξ belongs to $\mathcal{M}(\Omega)$ and

$$\Omega \xi = \mu + i s \xi, \quad \Omega \xi - i s \xi = \mu,$$

because of our initial assumption at the beginning of this paragraph.

Returning now to the number system $\mathcal{M}$ which is not necessarily commutative, let us consider a self-adjoint operator Ω having a bounded reciprocal $(\Omega - wI)^{-1}$ for some non-real $w = s + v$ ($v \neq 0$). Suppose that $\{\xi_n\}$ converges strongly to μ where the ξ 's are from $\mathcal{M}(\Omega)$ and μ lies in $\mathcal{M}$. Furthermore, assume that $\{\Omega \xi_n\}$ converges strongly to μ_* . Then

$$\lim_{n \rightarrow \infty} \xi_n = \lim_{n \rightarrow \infty} (\Omega - wI)^{-1} \xi_n = \xi$$

exists. Owing to the existence of the reciprocal there is an η in $\mathcal{M}_1 = \mathcal{M}(\Omega)$ such that

$$(4) \quad (\Omega - wI)^{-1} \xi = \eta, \quad (\Omega - wI) \eta = \xi, \\ (\Omega - wI)^{-1} (\xi_n - \xi) = \xi_n - \eta.$$

We know from (2) that

$$\|\xi_n - \eta\| \leq |1/v| \cdot \|\xi_n - \xi\|.$$

But $\{\xi_n\}$ converges and hence $\{\xi_n\}$ converges strongly to η .

On account of the unicity of the limit, we must have $\eta = \mu$, so that μ belongs to $\mathcal{M}_1$ and $\Omega \mu = \Omega \eta$. Now from (1) and (4)

$$\|(\Omega - sI)(\xi_n - \eta)\| \leq \|\xi_n - \xi\|,$$

hence

$$\mu_* - s\mu = \lim_{n \rightarrow \infty} (\Omega - sI) \xi_n = \Omega \eta - s\eta = \Omega \mu - s\mu.$$

Thus, $\mu^* = \Omega\mu$ when $\xi_n \rightarrow \mu$, $\Omega\xi_n \rightarrow \mu^*$.

We can now state the preceding result as follows.

THEOREM 1. A self-adjoint operator is always closed.

2. Resolvents and separations

Let Ω be a hermitian operator with $\mathcal{M}_1 \equiv \mathcal{M}(\Omega)$ dense on $\mathcal{M}$, $\mathcal{M}_2 \equiv \mathcal{M}_1 \times \Omega \mathcal{M}_1$. Define $\Omega_a \equiv \Omega - aI$ for a real or non-real number a . Then $\Gamma_a \equiv \Omega_a^{-1}$ is called the resolvent of Ω at a whenever the reciprocal exists as a bounded operator on $\mathcal{M}$.

We note that

$$\mathcal{M}(\Omega_a) = \mathcal{M}_1, \quad \Gamma_a \mathcal{M} \subset \mathcal{M}_1.$$

Moreover, Ω_a, Γ_a are linear with respect to real numbers. If ξ lies in $\mathcal{M}_1$ then put $\mu \equiv \Omega_a \xi$ and there is a μ with $\Gamma_a \mu = \xi$. Thus we have

$$\mathcal{M}(\Omega_a) = \mathcal{M}_1 = \Gamma_a \mathcal{M},$$

and ξ belongs to $\mathcal{M}_1$ when and only when it is of the form $\xi = \Gamma_a \mu$. For ξ in $\mathcal{M}_1$

$$\Gamma_a \xi = \zeta, \quad \xi = \Omega_a \Gamma_a \xi = \Omega_a \zeta.$$

As $\Omega_a \zeta = \xi$, $\Omega \zeta = \Omega_a \zeta + a \zeta$ must be in $\mathcal{M}_1$. Consequently Γ_a is on $\mathcal{M}_1$ to $\mathcal{M}_2$.

Consider $\Omega_a \Omega_{\bar{a}}$. Obviously, it is on $\mathcal{M}_2$ to $\mathcal{M}$. Now

$$\begin{aligned} \Omega_a \Omega_{\bar{a}} &= [(\Omega - sI) - vI][(\Omega - sI) + vI] \\ &= (\Omega - sI)^2 - v(\Omega - sI)(vI) - vI(vI) = (\Omega - sI)^2 + |v|^2 I. \end{aligned}$$

Similarly

$$\Omega_{\bar{a}} \Omega_a = (\Omega - sI)^2 + |-v|^2 I.$$

Introduce the operators

$$\pi_0 \equiv \Omega_{\bar{a}} \Omega_a = \Omega_{\bar{a}} \Omega_a, \quad \pi \equiv \Gamma_{\bar{a}} \Gamma_a.$$

Then π_0 is positive hermitian, π is on $\mathcal{M}$ to $\mathcal{M}_2$, and

$$\pi \pi_0 = \Gamma_{\bar{a}} \Gamma_{\bar{a}} \Omega_{\bar{a}} \Omega_a = \Gamma_{\bar{a}} \Omega_a = I \text{ as on } \mathcal{M}_2,$$

$$\pi_0 \pi = \Omega_{\bar{a}} \Omega_a \Gamma_{\bar{a}} \Gamma_a = \Omega_{\bar{a}} \Gamma_a = I \text{ as on } \mathcal{M}.$$

It follows that π_0 is properly positive, for, if $\pi_0 \xi = 0$,

$$\xi = \pi \pi_0 \xi = \pi 0 = 0.$$

Also π is positive hermitian, since

$$(\pi \mu, \xi) = (\pi \mu, \pi_0(\pi \xi)) = (\pi_0(\pi \mu), \pi \xi) = (\mu, \pi \xi),$$

$$(\pi \mu, \mu) = ((\pi \mu), \pi_0(\pi \mu))$$

As $\Gamma_{\bar{a}} \Gamma_a$ is on $\mathcal{M}$ to $\mathcal{M}_2$, $\pi_0 \Gamma_{\bar{a}} \Gamma_a$ exists on $\mathcal{M}$ and

$$\pi_0 \Gamma_{\bar{a}} \Gamma_a = \Omega_{\bar{a}} \Omega_a \Gamma_{\bar{a}} \Gamma_a = I \text{ as on } \mathcal{M}.$$

So we find

$$\pi_0(\pi - \Gamma_{\bar{a}} \Gamma_a) = 0,$$

which shows that

$$\Gamma_a \Gamma_{\bar{a}} = \pi = \Gamma_{\bar{a}} \Gamma_a,$$

because π_0 is properly positive.

Further, the next equality holds.

$$\Gamma_a + \Gamma_{\bar{a}} = 2 \Gamma_{\bar{a}} \Omega \Gamma_a.$$

Owing to the fact that Γ_a is linear with respect to real numbers, we have

$$\Gamma_a = \Gamma_{\bar{a}} \Omega_{\bar{a}} \Gamma_a, \quad \Gamma_{\bar{a}} = \Gamma_{\bar{a}} \Omega_a \Gamma_a.$$

Then it is not difficult to calculate

$$\begin{aligned} \Gamma_a + \Gamma_{\bar{a}} &= \Gamma_{\bar{a}} \Omega_{\bar{a}} \Gamma_a + \Gamma_{\bar{a}} \Omega_a \Gamma_a = \Gamma_{\bar{a}} [(\Omega - sI) - vI] \Gamma_a + \\ &\quad \Gamma_{\bar{a}} [(\Omega - sI) + vI] \Gamma_a \\ &= (\Gamma_{\bar{a}} - s\Gamma_{\bar{a}}) \Gamma_a - \Gamma_{\bar{a}} vI \Gamma_a + (\Gamma_{\bar{a}} \Omega - s\Gamma_{\bar{a}}) \Gamma_a + \Gamma_{\bar{a}} vI \Gamma_a \\ &= 2\Gamma_{\bar{a}} (\Omega - sI) \Gamma_a, \quad a \equiv s + v. \end{aligned}$$

Let Ψ be a bounded linear operator on $\mathcal{M}$. Suppose that Ψ, Ω are permutable on $\mathcal{M}_1$, i.e.,

$$\Psi \mathcal{M}_1 \subset \mathcal{M}_1, \quad \Psi \Omega = \Omega \Psi \quad \text{as on } \mathcal{M}_1.$$

Now Ω_a is on $\mathcal{M}_1$ so that $\Psi \Omega_a, \Omega_a \Psi$ are also on $\mathcal{M}_1$. But

$$\Psi \Omega_a = \Psi \Omega - a \Psi = \Omega \Psi - a \Psi, \quad \Omega_a \Psi = \Omega \Psi - a \Psi.$$

We have Γ_a on $\mathcal{M}$ to $\mathcal{M}_1$ and hence $\Gamma_a \Psi, \Psi \Gamma_a$ both on $\mathcal{M}$ to $\mathcal{M}_1$. As Ω_a, Ψ are permutable on $\mathcal{M}_1$,

$$\Psi \Omega_a \Gamma_a = \Omega_a \Psi \Gamma_a \quad \text{as on } \mathcal{M},$$

which gives

$$\Gamma_a \Psi = \Gamma_a \Psi (\Omega_a \Gamma_a) = \Gamma_a \Omega_a \Psi \Gamma_a = \Psi \Gamma_a \quad \text{as on } \mathcal{M}.$$

Conversely, suppose that Ψ, Γ_a are permutable on $\mathcal{M}$. Then Ψ is on $\mathcal{M}_1$ to $\mathcal{M}_1$ as Γ_a is on $\mathcal{M}$ to $\mathcal{M}_1$. Moreover, $\Omega_a \Psi, \Psi \Omega_a$ are on $\mathcal{M}_1$. From the equalities

$$\Psi \Omega_a \Gamma_a = \Psi = \Omega_a \Gamma_a \Psi = \Omega_a \Psi \Gamma_a$$

we find

$$\Psi \Omega_a = \Psi \Omega_a \Gamma_a \Omega_a = \Omega_a \Psi \Gamma_a \Omega_a = \Omega_a \Psi \quad \text{as on } \mathcal{M}_1.$$

This shows that $\mathcal{F}$, Ω are permutable on $\mathcal{M}_1$.

Finally we show that Γ_a , Ω are permutable on $\mathcal{M}_1$. Consider the equation $\Omega_a \zeta = \mu$. Then

$$a\mu = a\Omega_a \zeta = a\Omega_s \zeta - a v \zeta = \Omega_s(a \zeta) - v(a \zeta) = \Omega_a(a \zeta),$$

since $a \equiv s + v$, $av = va$. Thus $\Gamma_a(a\mu) = a\Gamma_a\mu$. We know that

$$\Gamma_a \Omega_a = \Omega_a \Gamma_a \text{ as on } \mathcal{M}_1.$$

Hence

$$\Gamma_a \Omega - a \Gamma_a = \Gamma_a \Omega_a = \Omega_a \Gamma_a = \Omega \Gamma_a - a \Gamma_a,$$

which leads directly to

$$\Gamma_a \Omega = \Omega \Gamma_a \text{ as on } \mathcal{M}_1.$$

In fact, $\Gamma_a \Omega$ is bounded, since $\Gamma_a \Omega = \Gamma_a \Omega_a + a \Gamma_a$.

Summarizing we have the next theorem.

THEOREM 2. Let Ω be a hermitian operator with $\mathcal{M}(\Omega)$ dense on $\mathcal{M}$, $\mathcal{M}_1 \equiv \mathcal{M}(\Omega)$, $\mathcal{M}_2 \equiv \mathcal{M}_1 \times \Omega \mathcal{M}_1$. Suppose that $\Omega_{\bar{a}} \equiv \Omega - aI$, $\Omega_{\bar{a}} = \Omega - \bar{a}I$ have bounded reciprocals Γ_a , $\Gamma_{\bar{a}}$ on $\mathcal{M}$ for a real or non-real a . Then

(5) ζ belongs to $\mathcal{M}_1$ if and only if it is of the form $\zeta = \Gamma_a \mu$ ($\zeta = \Gamma_{\bar{a}} \zeta$); Γ_a , $\Gamma_{\bar{a}}$ are on $\mathcal{M}_1$ to $\mathcal{M}_2$, and

$$\mathcal{M}_1 = \Gamma_a \mathcal{M} = \Gamma_{\bar{a}} \mathcal{M};$$

(6) the operator $\Gamma_{\bar{a}} \Gamma_a = \Gamma_a \Gamma_{\bar{a}}$ is on $\mathcal{M}$ to $\mathcal{M}_2$ and properly positive hermitian;

$$(7) \quad \Gamma_{\bar{a}} + \Gamma_a = 2 \Omega \Gamma_{\bar{a}} \Gamma_a = 2 \Gamma_{\bar{a}} \Omega \Gamma_a;$$

(8) Ω , Γ_a ($\Gamma_{\bar{a}}$) are permutable and $\Gamma_a \Omega$ ($\Gamma_{\bar{a}} \Omega$) is bounded. a bounded linear operator $\mathcal{F}$ on $\mathcal{M}$ is permutable with Ω when and only when it permutes with Γ_a ($\Gamma_{\bar{a}}$).

Now suppose that Ω is self-adjoint so that there is a non-real w for which the resolvents (bounded) Γ_w , $\Gamma_{\bar{w}}$ exist on $\mathcal{M}$. Define¹

$$(8') \quad \Phi_t \equiv \Omega_t \Gamma_{\bar{w}} \Gamma_w.$$

Then Φ_t is on $\mathcal{M}$ to $\mathcal{M}_1$, on $\mathcal{M}_1$ to $\mathcal{M}_2$, and is bounded, since Ω is on $\mathcal{M}_2$ to $\mathcal{M}_1$ and $\Omega_t \Gamma_{\bar{w}}$, Γ_w are bounded. It is also hermitian. Consider ξ in $\mathcal{M}_1$, ξ in $\mathcal{M}$. Clearly,

$$(\xi, \Phi_t \xi) = (\Omega_t \xi, \Gamma_{\bar{w}} \Gamma_w \xi) = (\Gamma_{\bar{w}} \Gamma_w \Omega_t \xi, \xi) = (\Phi_t \xi, \xi).$$

For any μ in $\mathcal{M}$,

$$\mu = \lim_{n \rightarrow \infty} \xi_n, \quad (\mu, \Phi_t \xi) = \lim_{n \rightarrow \infty} (\xi_n, \Phi_t \xi) = \lim_{n \rightarrow \infty} (\Phi_t \xi_n, \xi) = (\Phi_t \mu, \xi)$$

and hence Φ_t is hermitian on $\mathcal{M}$. Moreover, a bounded linear operator Ψ permutable with Ω on $\mathcal{M}_1$ is also permutable with Φ_t , because

$$\Psi \Omega_t \Gamma_{\bar{w}} \Gamma_w = \Omega_t \Psi \Gamma_{\bar{w}} \Gamma_w = \Omega_t \Gamma_{\bar{w}} \Gamma_w \Psi$$

as Ψ must permute with $\Gamma_{\bar{w}}$ and with Γ_w . In fact, Φ_t is permutable with Γ_w :

$$\Gamma_w \Phi_t = \Gamma_w \Omega_t \Gamma_{\bar{w}} \Gamma_w = \Omega_t \Gamma_w \Gamma_{\bar{w}} \Gamma_w = \Omega_t \Gamma_{\bar{w}} \Gamma_w \Gamma_w = \Phi_t \Gamma_w.$$

This implies that Γ_w permutes with operatorial polynomials $P(\Phi_t)$ and consequently with functions $F(\Phi_t)$. So, Γ_w and

$$S_t \equiv E_0(\Phi_t)$$

are permutable, hence S_t is on $\mathcal{M}_1$ to $\mathcal{M}_1$. As Φ_t , S_t both permute with Γ_w , they permute with Ω . Also Ψ permutes with S_t as it is permutable with Φ_t . In view of the equality

¹Cf. Riesz's Szeged paper.

$$(S_t \Omega_t \xi, \xi) = (\Omega_t \xi, S_t \xi) = (\xi, \Omega_t S_t \xi)$$

$S_t \Omega_t$ is hermitian on $\mathcal{M}_1$. Now

$$\begin{aligned} S_t \Omega_t &= \Omega_w \Gamma_w S_t \Omega_t = \Omega_w \Omega_t \Gamma_w S_t = \Omega_w (\Omega_t \Gamma_w \Gamma_w \Omega_w) S_t \\ &= \Omega_w (\Phi_t \Omega_w) S_t = \Omega_w \Omega_w \Phi_t S_t = (\Omega^2 + |w|^2 I) \Phi_t S_t \text{ as on } \mathcal{M}_1. \end{aligned}$$

From this we derive immediately

$$\begin{aligned} (S_t \Omega_t \xi, \xi) &= (\Omega^2 \Phi_t S_t \xi, \xi) + (|w|^2 \Phi_t S_t \xi, \xi) \\ &= (\Phi_t S_t (\Omega \xi), \Omega \xi) + (\Phi_t S_t (|w| \xi), |w| \xi) \\ &\leq 0, \end{aligned}$$

since $\Phi_t S_t = \Phi_t E_0(\Phi_t)$ is a negative operator by the theorem on separations in Chapter III. Similarly, we can prove that

$$(I - S_t) \Omega_t = (\Omega^2 + |w|^2 I) \Phi_t (I - S_t)$$

is a positive hermitian operator. Further, if $\Omega_t \xi = 0$ then

$$\Phi_t \xi = \Omega_t \Gamma_w \Gamma_w \xi = \Gamma_w \Gamma_w \Omega_t \xi = 0,$$

and we have $S_t \xi = 0$, again by the previous separation theorem.

Thus we finally arrive at

THEOREM 3. Given a bounded or unbounded self-adjoint operator Ω . Then to each real number t ($-\infty < t < +\infty$) there corresponds a separation operator S_t such that

(9) S_t is a projection operator,

(10) Ω , S_t are permutable and any bounded linear operator on $\mathcal{M}$ permutable with Ω permutes with S_t ,

(11) $S_t(\Omega - tI)$ is hermitian on $\mathcal{M}_1$, and $S_t(\Omega - tI)$, $(I - S_t)(\Omega - tI)$ are respectively negative and positive,

(12) if $\Omega \xi = t \xi$ then $S_t \xi = 0$.

Here we should remark that the above mentioned family of separations may be called a left system. However, it causes no new difficulty to show that there is also a right system S_t' which fulfills all the conditions (9)-(11). In lieu of (12) it satisfies

$$(12') \text{ if } \Omega\{\} = t\{\} \text{ then } S_t'\{\} = \{\}.$$

And, parallel to the left continuity of S_t (to be proved later), S_t' has right continuity.

The next theorem settles the unicity question of the separations.

THEOREM 4. For a self-adjoint operator there can be one and only one separation operator, satisfying conditions (9)-(12), at a given point on the real axis.

Consider two separations at t : S_t' , S_t'' . Let $W_t' = I - S_t'$, $W_t'' = I - S_t''$. By (10) these are all projection operators mutually permutable. We know that

$$S_t'W_t' = Z, \quad S_t''W_t'' = Z,$$

$$\Omega_{tW_t'}S_t'' = \Omega_{tW_t''}W_t'S_t'' = W_t'\Omega_{tW_t'}S_t''.$$

Now for $\{\}$ in $\mathcal{H}_1$

$$(\Omega_{tW_t'}S_t''\{\}, \{\}) = (S_t''\Omega_{tW_t'}S_t''\{\}, \{\}) = (\Omega_{tW_t'}(S_t''\{\}), S_t''\{\}) \\ \geq 0.$$

On the other hand, because of (11),

$$(\Omega_{tW_t'}S_t''\{\}, \{\}) = (W_t'\Omega_{tW_t'}S_t''\{\}, \{\}) = (\Omega_{tS_t''}(W_t'\{\}), W_t'\{\}) \\ \leq 0,$$

which with the previous inequality shows that $(\Omega_{tW_t'}S_t''\{\}, \{\}) = 0$

for every ξ in $\mathcal{M}_1$. Polarizing the left member of this equality we have $(\Omega_t W_t S_t'' \xi, \xi) = 0$ for any pair ξ, ζ in $\mathcal{M}_1$. For $\mu \equiv \Omega_t W_t S_t'' \xi$ there is a sequence ξ_n with

$$\lim_{n \rightarrow \infty} \xi_n = \mu,$$

and hence

$$\|\mu\|^2 = \lim_{n \rightarrow \infty} (\mu, \xi_n) = 0.$$

Thus

$$\Omega_t W_t S_t'' \xi = \mu = 0, \quad \Omega_t W_t S_t'' = Z.$$

Put $\eta \equiv W_t S_t'' \xi$. Then $\Omega_t \eta = 0$ so that $S_t'' \eta = 0$ by (12).

It follows that

$$0 = S_t''(W_t S_t'' \xi) = W_t S_t'' \xi, \quad W_t S_t'' = Z.$$

The last equality gives

$$S_t'' = (W_t' + S_t') S_t'' = S_t' S_t''.$$

Similarly $S_t' = S_t'' S_t'$. On account of the permutability we secure then $S_t' = S_t''$, which establishes the theorem.

3. Resolutions of the identity

A continuous family of projections $\{R_t\}$ $(-\infty < t < \infty)$ is called a resolution of the identity in case

(13) the projections are permutable,

(14) $R_s R_t = R_s$ for $s \leq t$,

(15) R_t is unilaterally continuous in the strong operatorial sense,

(16) $R_t \rightarrow Z$ or I as $t \rightarrow -\infty$ or $+\infty$.

We note that for any partition π of $(-\infty, +\infty)$

$$\sum_{\Delta} \Delta R_t = I, \quad \sum_{\Delta} (\Delta R_t) \mu = \mu,$$

and hence we have

$$\sum_{\Delta}^{\pi} (\xi, \Delta R_t \mu) = (\xi, \mu).$$

In consequence of (13), (14),

$$(\Delta R_t)^2 = \Delta R_t,$$

and

$$(\mu, \mu) = \sum_{\Delta}^{\pi} (\mu, \Delta R_t \mu) = \sum_{\Delta}^{\pi} \|\Delta R_t \mu\|^2.$$

Moreover,

$$(\xi, \Delta R_t \mu) = (\Delta R_t \xi, \Delta R_t \mu),$$

from which follows

$$\begin{aligned} |(\xi, \Delta R_t \mu)| &\leq \sum_{\Delta}^{\pi} \|\Delta R_t \xi\| \cdot \|\Delta R_t \mu\| \\ &\leq \left(\sum_{\Delta}^{\pi} \|\Delta R_t \xi\|^2 \right)^{1/2} \cdot \left(\sum_{\Delta}^{\pi} \|\Delta R_t \mu\|^2 \right)^{1/2} \\ &\leq (\xi, \xi)^{1/2} \cdot (\mu, \mu)^{1/2} \\ &\leq \|\xi\| \cdot \|\mu\|. \end{aligned}$$

Consequently R_t is a function of limited variation in the operational sense. Now as a function of limited variation $(R_t \xi, \mu)$ has unilateral limits, which are bounded, for,

$$\begin{aligned} |(R_{t \pm e} \xi, \mu)| &\leq \|\xi\| \cdot \|\mu\| \\ |(R_{t \pm 0} \xi, \mu)| &= \lim_{e=0} |(R_{t \pm e} \xi, \mu)| \leq \|\xi\| \cdot \|\mu\|. \end{aligned}$$

Since $R_{t \pm e}$ is positive hermitian, $R_{t \pm 0}$ is also positive hermitian. Moreover,

$$R_{t \pm e} R_{t \pm 0} = \lim_{d=0} R_{t \pm e} R_{t \pm d} = \lim_{d=0} \begin{Bmatrix} R_{t+d} \\ R_{t-e} \end{Bmatrix} = \begin{Bmatrix} R_{t+0} \\ R_{t-e} \end{Bmatrix},$$

from which we conclude that

$$R_{t+0}R_{t+0} = \lim_{e=0} R_{t+e}R_{t+0} = R_{t+0},$$

$$R_{t-0}R_{t-0} = \lim_{e=0} R_{t-e}R_{t-0} = \lim_{e=0} R_{t-e} = R_{t-0}.$$

Thus $R_{t\pm 0}$ is a projection operator. So, we have the following

THEOREM 5. Every resolution of the identity is a function of limited variation in the operatorial sense. It has unilateral limits which are again projection operators.

By virtue of this theorem we can now form Stieltjes integrals with respect to the integrator R_t . Let us introduce an operator $\mathfrak{F}$ as follows. The range of definition of $\mathfrak{F}$, $\mathcal{M}(\mathfrak{F})$ is the set of all ξ 's such that the next limit exists:

$$\int_{-\infty}^{+\infty} t dR_t \xi \equiv \lim_{N \rightarrow \infty} \sum_{N \neq 0} \frac{\pi}{\Delta} t_{\Delta} \Delta R_t \xi,$$

where t_{Δ} is any finite value in the interval Δ . And we put

$$(17) \quad \mathfrak{F}\xi \equiv \int_{-\infty}^{+\infty} t dR_t \xi.$$

We note in passing that, because of (13) and (14),

$$(18) \quad \Delta_1 R_t \cdot \Delta_2 R_t = \Delta_{12} R_t,$$

where Δ_{12} is the common portion of the intervals Δ_1, Δ_2 . In particular, $\Delta_1 R_t, \Delta_2 R_t$ are orthogonal when Δ_1, Δ_2 are non-overlapping. Consequently, for any partition π of $(-\infty, +\infty)$

$$(19) \quad \sum_{\Delta} \|\Delta R_t \mu\|^2 = \|\mu\|^2, \quad \sum_{\Delta} \Delta R_t \mu = \mu,$$

by (16). From (18) it follows also that for $\Delta \equiv (r, s)$

$$(20) \quad \begin{aligned} R_t \Delta R_t &= \Delta R_t, & t &\geq s, \\ R_t \Delta R_t &= Z, & t &\leq r. \end{aligned}$$

Consequently $\Delta \eta \equiv \Delta R_t \eta$ belongs to $\mathcal{M}(\mathfrak{F})$, and

$$\mathfrak{F}\Delta\eta \equiv \int_{-\infty}^{+\infty} t dR_t \Delta\eta = \int_{\Delta} t dR_t \gamma.$$

The obvious equality

$$\Delta\eta = \int_{\Delta} 1 dR_t \gamma$$

then gives

$$\mathfrak{F}\Delta\eta - t_{\Delta}\Delta\eta = \int_{\Delta} (t - t_{\Delta}) dR_t \gamma.$$

The norm of this vector-valued integral is

$$\begin{aligned} \left(\int_{\Delta} (t - t_{\Delta}) dR_t \gamma, \int_{\Delta} (t - t_{\Delta}) dR_t \gamma \right) &= \lim_{N\pi_{\Delta}=0}^L \left(\sum_{\delta}^{\pi_{\Delta}} (t - t_{\Delta}) \delta R_t \gamma, \sum_{\delta}^{\pi_{\Delta}} (t - t_{\Delta}) \delta R_t \gamma \right) \\ &= \lim_{N\pi_{\Delta}=0}^L \sum_{\delta}^{\pi_{\Delta}} ((t - t_{\Delta}) \delta R_t \gamma, (t - t_{\Delta}) \delta R_t \gamma) \\ &\quad \text{by (18)} \\ &= \lim_{N\pi_{\Delta}=0}^L \sum_{\delta}^{\pi_{\Delta}} ((t - t_{\Delta}) \delta^2 R_t \gamma, \gamma) \text{ again by (18)} \\ &= \lim_{N\pi_{\Delta}=0}^L \sum_{\delta}^{\pi_{\Delta}} (t - t_{\Delta}) \delta^2 \delta(R_t \gamma, \gamma) \\ &= \int_{\Delta} (t - t_{\Delta})^2 d(R_t \gamma, \gamma) \leq (N\pi)^2 \int_{\Delta} 1 d(R_t \gamma, \gamma), \end{aligned}$$

where π_{Δ} is a partition of Δ . So,

$$(21) \quad \left\| \int_{\Delta} (t - t_{\Delta}) dR_t \gamma \right\|^2 = \left\| \mathfrak{F}\Delta\eta - t_{\Delta}\Delta\eta \right\|^2 \leq (N\pi)^2 \|\Delta\eta\|^2,$$

which gives at once, for any partition π of $(-\infty, +\infty)$,

$$(22) \quad \sum_{\Delta}^{\pi} \left\| \mathfrak{F}\Delta\eta - t_{\Delta}\Delta\eta \right\|^2 \leq (N\pi)^2 \|\gamma\|^2,$$

when we recall (19).

If ξ is in $\mathfrak{M}(\mathfrak{F})$ so that the limit in (17) exists.

Then

$$\begin{aligned} \|\mathfrak{F}\xi\|^2 &= \left(\lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta R_t \xi, \lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta R_t \xi \right) \\ &= \lim_{N\pi=0}^L \sum_{\Delta}^{\pi} (t_{\Delta} \Delta R_t \xi, t_{\Delta} \Delta R_t \xi) \\ &\quad \lim_{N\pi=0}^L \sum_{\Delta}^{\pi} (t_{\Delta} \Delta R_t \xi, \xi) \end{aligned}$$

$$\begin{aligned}
 &= \lim_{N\pi=0} \sum_{\Delta}^{\pi} (t_{\Delta} \Delta R_t \xi, t_{\Delta} \Delta R_t \xi) \\
 &= \lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta}^2 \Delta (R_t \xi, \xi).
 \end{aligned}$$

This means that

$$\int_{-\infty}^{+\infty} t^2 d(R_t \xi, \xi) = \|\Psi \xi\|^2 < +\infty.$$

Also

$$(\Psi \xi, \mu) = \left(\lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta R_t \xi, \mu \right) = \lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta (R_t \xi, \mu)$$

which is precisely a Stieltjes integral, and we obtain

$$(\Psi \xi, \mu) = \int_{-\infty}^{+\infty} t d(R_t \xi, \mu).$$

Conversely, suppose that η is a vector for which

$$(23) \quad \int_{-\infty}^{+\infty} t^2 d(R_t \eta, \eta) < +\infty.$$

Then there is a $d > 0$ such that, for partitions π of $(-\infty, +\infty)$ with $N\pi \leq d$,

$$(24) \quad \sum_{\Delta}^{\pi} t_{\Delta}^2 \Delta (R_t \eta, \eta) = \sum_{\Delta}^{\pi} t_{\Delta}^2 \|\Delta R_t \eta\|^2 \equiv \sum_{-\infty < k < +\infty}^{\pi} t_k^2 \|\Delta_k R_t \eta\|^2 < \infty, t_k \equiv t_{\Delta_k}.$$

From the equalities

$$\begin{aligned}
 \left\| \sum_{|k| \leq n} t_k \Delta_k R_t \eta - \sum_{|k| \leq m} t_k \Delta_k R_t \eta \right\|^2 &= \left\| \sum_{m+1 \leq |k| \leq n} t_k \Delta_k R_t \eta \right\|^2 \\
 &= \sum_{m+1 \leq |k| \leq n} t_k^2 \|\Delta_k R_t \eta\|^2,
 \end{aligned}$$

when $m < n$, we see that

$$(25) \quad \sum_{\Delta}^{\pi} t_{\Delta} \Delta R_t \eta$$

exists as a strong limit by the Cauchy sufficient condition, since the infinite series in (24) converges to a finite sum. By the triangle inequalities we find

$$\begin{aligned}
 & \left| \left(\sum_{|k| \leq n} \frac{\pi}{\Delta_k} \|\Psi \Delta_k \gamma\|^2 \right)^{1/2} - \left(\sum_{|k| \leq n} t_k^2 \|\Delta_k \gamma\|^2 \right)^{1/2} \right| = \\
 & \left| \left\| \sum_{|k| \leq n} \frac{\pi}{\Delta_k} \Psi \Delta_k \gamma \right\| - \left\| \sum_{|k| \leq n} t_k \Delta_k \gamma \right\| \right| \leq \left\| \sum_{|k| \leq n} (\Psi \Delta_k \gamma - t_k \Delta_k \gamma) \right\| \\
 & = \left(\sum_{|k| \leq n} \|\Psi \Delta_k \gamma - t_k \Delta_k \gamma\|^2 \right)^{1/2} \leq N \pi \|\gamma\|, \quad \Delta \gamma \equiv \Delta R_t \gamma,
 \end{aligned}$$

from (22). Accordingly

$$(26) \quad \sum_{-\infty < k < \infty} \frac{\pi}{\Delta_k} \|\Psi \Delta_k \gamma\|^2 < \infty,$$

in virtue of (24). As

$$\left\| \sum_{|k| \leq n} \frac{\pi}{\Delta_k} \Psi \Delta_k \gamma - \sum_{|k| \leq m} \frac{\pi}{\Delta_k} \Psi \Delta_k \gamma \right\|^2 = \sum_{m+1 \leq k \leq n} \frac{\pi}{\Delta_k} \|\Psi \Delta_k \gamma\|^2,$$

the next series converges strongly:

$$(27) \quad \sum_{-\infty < k < \infty} \frac{\pi}{\Delta_k} \Psi \Delta_k \gamma \equiv \gamma_\pi = \frac{\pi}{\Delta} \Psi \Delta \gamma = \frac{\pi}{\Delta} \int_{\Delta} t \, dR_t \gamma,$$

where γ_π ordinarily depends on π . Consider an arbitrary partition π' of $(-\infty, +\infty)$ with $N\pi' \leq d$. The points in π and π' together form another partition π^* so that $\pi \subset \pi^*$, $\pi' \subset \pi^*$. Each interval of π or π' contains only a finite number of intervals from π^* . It is then obvious that

$$(28) \quad \gamma_\pi = \frac{\pi}{\Delta} \Psi \Delta \gamma = \frac{\pi^*}{\Delta} \Psi \Delta \gamma = \frac{\pi'}{\Delta} \Psi \Delta \gamma.$$

Thus γ_π is independent of any particular partition and we write $\gamma_\pi \equiv \zeta$. Because of (25) and (27),

$$\frac{\pi}{\Delta} \Psi \Delta \gamma - \frac{\pi}{\Delta} t_\Delta \Delta \gamma = \frac{\pi}{\Delta} \int_{\Delta} (t - t_\Delta) dR_t \gamma$$

exists as a strong limit. Further,

$$\begin{aligned}
 \left\| \frac{\pi}{\Delta} \int_{\Delta} (t - t_\Delta) dR_t \gamma \right\|^2 &= \frac{\pi}{\Delta} \left(\int_{\Delta} (t - t_\Delta) dR_t \gamma, \frac{\pi}{\Delta} \int_{\Delta} (t - t_\Delta) dR_t \gamma \right) \\
 &= \frac{\pi}{\Delta} \left\| \int_{\Delta} (t - t_\Delta) dR_t \gamma \right\|^2 \leq (N\pi)^2 \|\gamma\|^2
 \end{aligned}$$

by (21). This means simply that

$$\| \xi - \sum_{\Delta}^{\pi} t_{\Delta} \Delta \eta \| \leq N \pi \| \eta \| \leq \epsilon \| \eta \|.$$

So, we have finally

$$\xi = \lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta \eta = \int_{-\infty}^{+\infty} t \, dR_t \eta = \mathfrak{T} \eta.$$

In other words, a vector η satisfying (23) belongs to $\mathcal{M}(\mathfrak{T})$.

Concerning $\mathcal{M}(\mathfrak{T})$ we observe that it is linear. Now, for a partition π of $(-\infty, +\infty)$

$$(29) \quad \mu = \lim_{n=\infty} \sum_{|k| \leq n}^{\pi} \Delta_k R_t \mu \equiv \lim_{n=\infty} \mu \pi_n,$$

where μ is any element in $\mathcal{M}$. From (20)

$$\int_{\Delta_k} t^2 d(R_t \mu, \mu) = \int_{-\infty}^{+\infty} t^2 d(R_t \Delta_k R_t \mu, \Delta_k R_t \mu),$$

and hence by (18)

$$\int_{-\infty}^{+\infty} t^2 d(R_t \mu \pi_n, \mu \pi_n) = \sum_{|k| \leq n}^{\pi} \int_{\Delta_k} t^2 d(R_t \mu, \mu) < \infty.$$

Thus $\mu \pi_n$ satisfies (23) and from what precedes it lies in $\mathcal{M}(\mathfrak{T})$.

Then (29) shows that every vector of $\mathcal{M}$ can be expressed as the strong limit of a sequence of vectors from $\mathcal{M}(\mathfrak{T})$. Hence $\mathcal{M}(\mathfrak{T})$ is dense on $\mathcal{M}$.

The operator $\mathfrak{T}$ defined in (17) is in fact self-adjoint.

Clearly it is linear and hermitian, and $\mathcal{M}(\mathfrak{T})$ is linear and dense on $\mathcal{M}$. Consider a non-real number $w = s + v$, $v \neq 0$. The function is certainly bounded over the entire infinite interval. Though not real-valued it is nevertheless uniformly continuous in t :

$$|1/(t_2 - w) - 1/(t_1 - w)| \leq |t_2 - t_1| \cdot |1/v|^2.$$

Let π' , π'' be two partitions of $(-\infty, +\infty)$ with their norms less than ϵ . Superimpose one upon the other and we obtain π^* such that $\pi' \subset \pi^*$, $\pi'' \subset \pi^*$. Then, as in the case of numerical integration,

$$\begin{aligned} \left\| \sum_{\Delta} F(t_{\Delta}) \Delta R_t \mu - \sum_{\Delta} F(t_{\Delta}) \Delta R_t \mu \right\|^2 &= \left\| \sum_{\Delta} (F(t_{\Delta}') - F(t_{\Delta}'')) \Delta R_t \mu \right\|^2 \\ &= \sum_{\Delta} |F(t_{\Delta}') - F(t_{\Delta}'')|^2 (\Delta R_t \mu, \mu) \\ &\leq \sum_{\Delta} (2|1/v|^3 \cdot |t_{\Delta}' - t_{\Delta}''|)^2 (\Delta R_t \mu, \mu) \\ &\leq 4|1/v|^6 (N\pi')^2 \|\mu\|^2, \end{aligned}$$

where $F(t)$ stands for $1/(t - w)$. Thus $\left\| \sum_{\Delta} \pi' - \sum_{\Delta} \pi'' \right\| \rightarrow 0$ as the norms of π' , π'' simultaneously approach zero. It follows that the next operator exists for any μ in π :

$$(30) \quad (\Psi - wI) * \mu \equiv \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu \equiv \lim_{N\pi=0} \sum_{\Delta} \pi (1/(t - w)) \Delta R_t \mu.$$

To discuss the relations of this with $\Psi - wI$ we first show that

$$(31) \quad (\Psi - wI) * \Psi \xi = \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \xi,$$

for every ξ in $\pi(\Psi)$, or, what is the same,

$$\int_{-\infty}^{+\infty} (1/(t - w)) (\Psi - tI) dR_t \xi = 0,$$

which is the limit (as the norm of π becomes indefinitely small) of

$$\gamma_{\pi} \equiv \sum_{\Delta} \frac{1}{t_{\Delta} - w} (\Psi - t_{\Delta} I) \Delta R_t \xi = \sum_{\Delta} \frac{1}{t_{\Delta} - w} (\Psi \Delta \xi - t_{\Delta} \Delta \xi), \quad \Delta \xi \equiv \Delta R_t \xi.$$

On account of (18) and (21) we find

$$\|\gamma_{\pi}\|^2 = \sum_{\Delta} |1/(t - w)|^2 \|\Psi \Delta \xi - t_{\Delta} \Delta \xi\|^2 \leq |1/v|^2 (N\pi)^2 \|\xi\|^2.$$

Consequently γ_{π} converges strongly to 0 as $N\pi \rightarrow 0$, and (31) is established. This equality implies further

$$(32) \quad (\mathfrak{L} - wI)^*(\mathfrak{L} - wI) \left\{ \right\} = \int_{-\infty}^{+\infty} ((t - w)/(t - w)) dR_t \left\{ \right\} = \left\{ \right\},$$

so that $(\mathfrak{L} - wI)^*$ is the left reciprocal of $\mathfrak{L} - wI$. On the other hand, we infer from (29) that

$$\|\mu_{\pi n} - \mu\| < \epsilon, \quad \mu_{\pi n} \equiv \sum_{|k| \leq n} \frac{\pi}{\Delta_k} \Delta_k \mu, \quad n > n_\epsilon.$$

This gives

$$\begin{aligned} \left\| \int_{-\infty}^{+\infty} (t/(t - w)) dR_t (\mu_{\pi n} - \mu) \right\|^2 &= \int_{-\infty}^{+\infty} |t/(t - w)|^2 d(R_t (\mu_{\pi n} - \mu), \mu_{\pi n} - \mu) \\ &\leq (1 + |w/v|)^2 \int_{-\infty}^{+\infty} 1 d(R_t (\mu_{\pi n} - \mu), \mu_{\pi n} - \mu) \\ &\leq (1 + |w/v|)^2 \cdot \|\mu_{\pi n} - \mu\|^2 \\ &\leq (1 + |w/v|)^2 \epsilon^2, \quad n > n_\epsilon. \end{aligned}$$

Now by (20),

$$\int_{\Delta_k} (t/(t - w)) dR_t \mu = \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \Delta_k \mu.$$

Clearly then

$$(33) \quad \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \int_{\Delta_k} (t/(t - w)) dR_t \mu = \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \mu.$$

But

$$\begin{aligned} \left\| t_k \int_{\Delta_k} (1/(t - w)) dR_t \mu - \int_{\Delta_k} (t/(t - w)) dR_t \mu \right\|^2 &= \\ &= \int_{\Delta_k} |(t_k - t)/(t - w)|^2 d(R_t \mu, \mu) \\ &\leq (N\pi)^2 \int_{\Delta_k} |1/(t - w)|^2 d(R_t \mu, \mu) \leq (N\pi)^2 |1/v|^2 \cdot \|\Delta_k \mu\|^2, \\ &\quad t_k \equiv t_{\Delta_k}, \end{aligned}$$

from which follows

$$(34) \quad \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \left\| \int_{\Delta_k} ((t_k - t)/(t - w)) dR_t \mu \right\|^2 \leq (N\pi)^2 |1/v|^2 \|\mu\|^2 < \infty.$$

In consequence,

$$\left\| \sum_{|k| \leq n} \frac{\pi}{\Delta_k} \int_{\Delta_k} \frac{t_k - t}{t - w} dR_t \mu - \sum_{|k| \leq m} \frac{\pi}{\Delta_k} \int_{\Delta_k} \frac{t_k - t}{t - w} dR_t \mu \right\|^2 = \sum_{m+1 \leq |k| \leq n} \frac{\pi}{\Delta_k} \left\| \int_{\Delta_k} \frac{t_k - t}{t - w} dR_t \mu \right\|^2$$

converges to zero as m and n simultaneously become infinite. Thus the series

$$\sum_{|k| < \infty} \frac{\pi}{\Delta_k} \int_{\Delta_k} ((t_k - t)/(t - w)) dR_t \mu$$

converges strongly. This with (33) shows that

$$(35) \quad \sum_{|k| < \infty} \frac{\pi}{\Delta_k} t_k \Delta_k R_t \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu = \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \int_{\Delta_k} (t_k/(t - w)) dR_t \mu$$

exists as a strong limit. Moreover,

$$\begin{aligned} & \left\| \sum_{|k| < \infty} \frac{\pi}{\Delta_k} t_k \Delta_k R_t \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu - \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \mu \right\|^2 \\ &= \left\| \sum_{|k| < \infty} \frac{\pi}{\Delta_k} t_k \Delta_k R_t \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu - \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \int_{\Delta_k} (t/(t - w)) dR_t \mu \right\|^2 \\ &= \left\| \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \int_{\Delta_k} ((t_k - t)/(t - w)) dR_t \mu \right\|^2 \\ &= \sum_{|k| < \infty} \frac{\pi}{\Delta_k} \left\| \int_{\Delta_k} ((t_k - t)/(t - w)) dR_t \mu \right\|^2 \leq (N\pi)^2 |1/v|^2 \|\mu\|^2, \end{aligned}$$

by (33), (35), and by (34). Hence

$$(36) \quad \int_{-\infty}^{+\infty} t dR_t \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu = \lim_{N \rightarrow 0} \frac{\pi}{\Delta} \sum_{\Delta} t_{\Delta} \Delta R_t \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \mu \\ = \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \mu.$$

The existence of the left member means that $(\Psi - wI)*\mu$ lies in $\mathcal{M}(\Psi)$ so that $(\Psi - wI)*$ is on $\mathcal{M}$ to $\mathcal{M}(\Psi)$. Then without difficulty we show that $(\Psi - wI)*$ is also the right reciprocal of $\Psi - wI$. For by (36),

$$\begin{aligned} \Psi(\Psi - wI)*\mu &= \int_{-\infty}^{+\infty} (t/(t - w)) dR_t \mu = \int_{-\infty}^{+\infty} ((t - w)/(t - w)) dR_t \mu \\ &\quad + \int_{-\infty}^{+\infty} (w/(t - w)) dR_t \mu = \mu + w(\Psi - wI)*\mu, \end{aligned}$$

and hence

$$(\Psi - wI)(\Psi - wI)*\mu = \mu,$$

which, in conjunction with (32), gives

$$P_w = (\Psi - wI)^*.$$

As R_t , $(\Psi - wI)^*$ are permutable, R_t permutes with P_w and consequently with Ω , by (8) of Theorem 2.

If ξ , ξ^* satisfy the relation

$$(\xi, \Psi \xi) = (\xi^*, \xi)$$

for every ξ of $\mathcal{M}(\Psi)$, then

$$(\Psi \Delta \xi, \xi) = (\Delta \xi, \Psi \xi) = (\xi, \Psi \Delta \xi) = (\xi^*, \Delta \xi) = (\Delta \xi^*, \xi),$$

so that $\Psi \Delta \xi = \Delta \xi^*$. But

$$\sum_{\Delta}^{\pi} \Delta \xi^* = \xi^*,$$

and therefore

$$\sum_{\Delta}^{\pi} \Psi \Delta \xi = \xi^*.$$

We have already seen that Ψ is self-adjoint. Then by Theorem 1 it is closed. This closure property of Ψ together with the fact that $\sum \Delta \xi = \xi$ shows that ξ belongs to $\mathcal{M}(\Psi)$ and $\Psi \xi = \xi^*$. In this way we are led to the equality

$$\Psi^* = \Psi.$$

We state these results in

THEOREM 6. Associated with a resolution of the identity $\{R_t\}$ there is an operator Ψ permutable with R_t , defined by the equation

$$\Psi \xi \equiv \int_{-\infty}^{+\infty} t \, dR_t \xi \equiv \lim_{N \rightarrow \infty} \sum_{\Delta}^{\pi} t_{\Delta} \Delta R_t \xi,$$

the independent range $\mathcal{M}(\Psi)$ being the set of vectors for each of which the last limit exists. In fact, $\mathcal{M}(\Psi)$ is linear, dense on $\mathcal{M}$; Ψ is self-adjoint, and $\Psi^* = \Psi$. The resolvent

$\Gamma_w \equiv (\Psi - wI)^{-1}$ exists for every non-real number w and

$$\Gamma_w = \int_{-\infty}^{+\infty} (1/(t - w)) dR_t \quad .$$

Moreover, ξ belongs to $\mathfrak{M}(\Psi)$ if and only if

$$\int_{-\infty}^{+\infty} t^2 d(R_t \xi, \xi) < +\infty \quad .$$

For ξ in $\mathfrak{M}(\Psi)$ we have

$$(\Psi \xi, \mu) = \int_{-\infty}^{+\infty} t \, d(R_t \xi, \mu), \quad \|\Psi \xi\|^2 = \int_{-\infty}^{+\infty} t^2 d(R_t \xi, \xi).$$

It may be recalled that the separation operators S_t form a continuous family of projections. Is this family a resolution of the identity? This question we proceed to answer.¹

First of all, the separations are mutually permutable because of (10) in Theorem 3, as each of them permutes with Ω . Next, $S_r S_t = S_r$ for $r \leq t$. In fact, we wish to show that $S_r(I - S_t) = Z$. This is trivial when $r = t$. Consider the case with $r < t$. Put $T = S_r(I - S_t) = (I - S_t)S_r$. Then T is positive hermitian, and we have

$$T^2 = T, \quad T = S_r(I - S_t) = S_r^2(I - S_t) = S_r T, \quad (I - S_t)T = T.$$

From Theorem 3,

$$\begin{aligned} (S_r \Omega_r T, \xi) &= (S_r \Omega_r T, T\xi) \leq 0 \leq ((I - S_t) \Omega_t T, T\xi) \\ &= ((I - S_t) \Omega_t T, \xi), \end{aligned}$$

ξ being in $\mathfrak{M}_1$, so that

$$S_r \Omega_r T \leq Z \leq (I - S_t) \Omega_t T \text{ on } \mathfrak{M}_1.$$

¹The proof is that of Riesz with occasional modifications.

As $S_r T = T$ and $(I - S_t)T = T$, the following inequalities hold:

$$\begin{aligned}\Omega_r T &= \Omega_r T T = \Omega_r T S_r T = S_r \Omega_r T \leq Z \quad \text{on } \mathcal{M}_1, \\ \Omega_t T &= \Omega_t (I - S_t) T = (I - S_t) \Omega_t T \geq Z \quad \text{on } \mathcal{M}_1.\end{aligned}$$

Subtracting the left members, we find

$$(t - r)T = \Omega_r T - \Omega_t T \leq Z \quad \text{on } \mathcal{M}_1.$$

In the limit, $T \leq Z$ on $\mathcal{M}$. But T is positive; it must then be the zero operator. Thus we have proved properties (13), (14) for $\{S_t\}$ as a resolution of the identity. For (16) it suffices to prove that S_t converges to the zero operator as t becomes $-\infty$, since the other case can be obtained readily by considering $I - S_t$ in place of S_t . Because of property (14),

$$\|S_t \mu\|^2 = (S_t \mu, \mu)$$

is a monotone increasing function of t , for,

$$S_t - S_r = S_t - 2S_r + S_r = S_t^2 - 2S_t S_r + S_r^2 = (S_t - S_r)^2 \geq Z.$$

Hence the limit of $\|S_t \mu\|^2$ as $t \rightarrow -\infty$ exists. For ξ in $\mathcal{M}_1$ and $t < 0$ we have

$$\begin{aligned}\|S_t \xi\|^2 &= (S_t \xi, \xi) = (1/t)(S_t \Omega \xi, \xi) - (1/t)(S_t \Omega_t \xi, \xi) \\ &\leq (1/t)(S_t \Omega \xi, \xi) \leq |1/t| \cdot \|S_t \Omega \xi\| \cdot \|\xi\| \leq |1/t| \cdot \|\Omega \xi\| \cdot \|\xi\|,\end{aligned}$$

as $S_t \Omega_t$ is negative and hence $(1/t)S_t \Omega_t$ is positive when $t < 0$. From this it follows immediately that $S_t \xi$ converges strongly to the zero vector as $t \rightarrow -\infty$. Consider now a μ in $\mathcal{M}$. Let ϵ be a non-zero positive number. Then there is a ξ_ϵ of $\mathcal{M}_1$ such that

$$\|\mu - \xi_\epsilon\| < \epsilon/2.$$

On the other hand, there exists a t_e with

$$\|S_t \xi_e\| < e/2, \quad t < t_e.$$

By virtue of the inequality

$$\begin{aligned} \|S_t \mu\| &= \|S_t(\mu - \xi_e) + S_t \xi_e\| \leq \|S_t(\mu - \xi_e)\| + \|S_t \xi_e\| \\ &\leq \|\mu - \xi_e\| + \|S_t \xi_e\|, \end{aligned}$$

we find

$$\|S_t \mu\| < e, \quad t < t_e,$$

so that $S_t \mu$ converges to the zero vector in the strong sense.

To show property (15) that S_t , as a function of t , is left continuous in the strong operatorial sense proceed as follows. Inasmuch as

$$\|S_t \mu - S_r \mu\|^2 = (S_t \mu, \mu) - (S_r \mu, \mu),$$

we need only consider the continuity of $(S_t \mu, \mu)$. Let r approach t from the left. Owing to the monotonicity of $(S_t \mu, \mu)$, the limit of $(S_r \mu, \mu)$ exists as $r \rightarrow t$, and consequently

$$\|S_s \mu - S_r \mu\|^2 = (S_s \mu, \mu) - (S_r \mu, \mu), \quad r < s < t,$$

converges to zero as r and s simultaneously approach t from the left. Since the space $\mathcal{M}$ is complete, the Cauchy sufficient condition shows that there is a ξ such that $S_r \mu$ converges strongly to ξ . So we have the operator

$$S_{t-0} \equiv \lim_{e \rightarrow 0} S_{t-e}, \quad e > 0.$$

It is obviously a projection permutable with every one in the family of separations. We are to show that $S_{t-0} = S_t$, or $\Delta_0 = Z$, where

$$\Delta_0 \equiv \lim_{e \rightarrow 0} \Delta_{t-e}, \quad \Delta_r \equiv S_t - S_r.$$

Evidently $\Delta_0 = S_t - S_{t-0}$. Applying Theorem 3,

$$(S_t \Omega_t \Delta_r \}, \Delta_r \}) \leq 0 \leq ((I - S_r) \Omega_r \Delta_r \}, \Delta_r \}),$$

which, on account of

$$(37) \quad \Delta_r^2 = \Delta_r, \quad S_t \Delta_r = (I - S_r) \Delta_r = \Delta_r,$$

gives

$$(38) \quad (\Omega_t \Delta_r \}, \Delta_r \}) \leq 0 \leq (\Omega_r \Delta_r \}, \Delta_r \}),$$

since Ω_t , S_r , S_t , and Δ_r are mutually permutable. Now for $r \leq s \leq t$ we have

$$(\Omega_s \Delta_r \}, \Delta_r \}) = (\Omega \Delta_r \}, \Delta_r \}) - s(\Delta_r \}, \Delta_r \}),$$

so that

$$r(\Delta_r \}, \Delta_r \}) \leq (\Omega \Delta_r \}, \Delta_r \}) \leq t(\Delta_r \}, \Delta_r \}),$$

by (38). Since Δ_r is positive,

$$\begin{aligned} (r - t)(\Delta_r \}, \Delta_r \}) &\leq (r - s)(\Delta_r \}, \Delta_r \}) \leq (\Omega \Delta_r \}, \Delta_r \}) \\ &\quad - s(\Delta_r \}, \Delta_r \}), \\ (\Omega \Delta_r \}, \Delta_r \}) - s(\Delta_r \}, \Delta_r \}) &\leq (t - s)(\Delta_r \}, \Delta_r \}) \\ &\leq (t - r)(\Delta_r \}, \Delta_r \}). \end{aligned}$$

Hence

$$-(t - r)(\Delta_r \}, \Delta_r \}) \leq (\Omega \Delta_r \}, \Delta_r \}) - s(\Delta_r \}, \Delta_r \}) \leq (t - r)(\Delta_r \}, \Delta_r \}),$$

or,

$$(39) \quad |(\Omega_s \Delta_r \}, \Delta_r \})| \leq (t - r)(\Delta_r \}, \Delta_r \}).$$

From the inequality $(\Delta_r \}, \Delta_r \}) \leq \|\Delta_r\|^2$ we see that

$$|(\Omega_s \Delta_r \}, \Delta_r \})| \leq (t - r)\|\Delta_r\|^2,$$

which shows that

$$0 \leq ((tI - \Omega) \Delta_r \xi, \xi) \leq (t - r) \|\xi\|^2,$$

$$0 \leq (\Omega_r \Delta_r \xi, \xi) \leq (t - r) \|\xi\|^2$$

In particular, $(tI - \Omega) \Delta_r$ is hermitian on $\mathcal{M}_1$, and

$$0 \leq ((tI - \Omega) \Delta_r \xi, \xi) \leq (t - r) \|\xi\|^2,$$

so that the absolute values of its upper and lower moduli do not exceed $t - r$, and hence $M((tI - \Omega) \Delta_r) \leq t - r$, by a theorem on moduli of hermitian operators in Chapter II. Then from the definition of modulus of an operator,

$$\|(tI - \Omega) \Delta_r \xi\| \leq (t - r) \|\xi\|.$$

Due to the permutability of Ω_t and Δ_r on $\mathcal{M}_1$,

$$\|\Delta_r \Omega \xi - t \Delta_r \xi\| \leq (t - r) \|\xi\|, \quad r < t,$$

from which it follows that $\Delta_r \Omega \xi$ converges strongly to $\Delta_0 \xi$, or $\Delta_0 \Omega \xi = t \Delta_0 \xi$. We know that Δ_r is permutable with Ω and hence with Γ_w . Consequently in the limit Δ_0 permutes with Γ_w . According to (8) of Theorem 2, Δ_0 is then permutable with Ω on $\mathcal{M}_1$ so that $\Omega \Delta_0 \xi = t \Delta_0 \xi$. Consider now $\eta \equiv \Delta_0 \xi$. Here we have $\Omega_t \eta = 0$, which, by (12) of Theorem 3, gives $S_t \eta = 0$. On the other hand, $\Delta_r S_t = \Delta_r$ from (37), and $\Delta_0 S_t = \Delta_0$ as $r \rightarrow t$. Thus,

$$\Delta_0 \xi = \Delta_0 S_t \xi = S_t \Delta_0 \xi = S_t \eta = 0,$$

and hence $\Delta_0 \xi = 0$ for every ξ of $\mathcal{M}_1$ which is dense on $\mathcal{M}$. Passing to the limit, we find $\Delta_0 \mu = 0$ for every μ of $\mathcal{M}$, or $\Delta_0 = Z$.

THEOREM 7. The separations of a self-adjoint operator form a resolution of the identity.

4. Canonical representation

For a self-adjoint operator Ω we have

$$(40) \quad \|\Omega \Delta\} - s \Delta\}\| \leq (t - r) \|\Delta\} \quad , \quad \Delta \equiv (r, t),$$

$$\Delta\} \equiv (S_t - S_r)\} \quad , \quad r \leq s \leq t,$$

which is an immediate consequence of (39) and a theorem on the modulus of a hermitian operator in Chapter II. Let η be $\Omega\}$. Consider a partition π of $(-\infty, +\infty)$. Due to the permutability of Ω with its separations,

$$\Delta \eta = \Delta \Omega\} = \Omega \Delta\}.$$

With the aid of (19) it is readily deduced that

$$(41) \quad \Omega\} = \eta = \sum_{\Delta} \frac{\pi}{\Delta} \Omega \Delta\}.$$

However (40) shows that

$$(42) \quad \sum_{\Delta} \|\Omega \Delta\} - t_{\Delta} \Delta\|\|^2 \leq (N\pi)^2 \|\}\|^2,$$

which yields the strong convergence of

$$\sum_{\Delta} (\Omega \Delta\} - t_{\Delta} \Delta\}).$$

By (41) and (42),

$$\|\Omega\} - \sum_{\Delta} t_{\Delta} \Delta\|\|^2 = \left\| \sum_{\Delta} \frac{\pi}{\Delta} (\Omega \Delta\} - t_{\Delta} \Delta\}) \right\|^2 = \sum_{\Delta} \|\Omega \Delta\} - t_{\Delta} \Delta\|\|^2 \leq (N\pi)^2 \|\}\|^2.$$

As $N\pi \rightarrow 0$ we find

$$\Omega\} = \lim_{N\pi \rightarrow 0} \sum_{\Delta} \frac{\pi}{\Delta} t_{\Delta} \Delta\} = \int_{-\infty}^{+\infty} t \, dS_t\}.$$

Using the results in Theorem 6 and Theorem 7, we secure

THEOREM 8. If S_t , $-\infty < t < +\infty$ are the separations of a self-adjoint operator Ω we have the following simultaneous representations:

$$\Omega = \int_{-\infty}^{+\infty} t \, dS_t \text{ as on } \mathcal{M}(\Omega), \quad I = \int_{-\infty}^{+\infty} 1 \, dS_t \text{ as on } \mathcal{M},$$

where the Stieltjes integrals are taken in the strong sense of norm. A necessary and sufficient condition for an element ξ to belong to the independent range of Ω is that

$$\int_{-\infty}^{+\infty} t^2 d(S_t \xi, \xi) < +\infty.$$

If ξ is in this range,

$$\|\Omega \xi\|^2 = \int_{-\infty}^{+\infty} t^2 d(S_t \xi, \xi), \quad (\Omega \xi, \mu) = \int_{-\infty}^{+\infty} t \, d(S_t \xi, \mu).$$

Suppose that two self-adjoint operators Ω_1, Ω_2 have the same separations, then by the second part of the previous theorem $\mathcal{M}(\Omega_1) = \mathcal{M}(\Omega_2)$ and consequently by the first part of the same theorem $\Omega_1 = \Omega_2$. Thus, we have

THEOREM 9. If two self-adjoint operators have the same separations then the operators themselves are equal.

Another question suggests itself. When two resolutions define the same operator, do the resolutions always coincide? More precisely, consider two left continuous resolutions R_t', R_t'' with

$$\int_{-\infty}^{+\infty} t \, dR_t' = \int_{-\infty}^{+\infty} t \, dR_t'' = \Omega.$$

We first show that

$$(43) \quad \Omega_s \xi = 0 \text{ implies } R_s' \xi = 0.$$

From $\Omega \xi = s \xi$ we derive

$$s^2 \|\xi\|^2 = \|\Omega \xi\|^2 = \int_{-\infty}^{+\infty} t^2 d(R_t' \xi, \xi).$$

Then because of

$$\|\xi\|^2 = \int_{-\infty}^{+\infty} 1 \, d(R_t' \xi, \xi),$$

we can write

$$(44) \quad \int_{-\infty}^{+\infty} (t-s)^2 d(R_t' \xi, \xi) = 0.$$

Let b_Δ be the minimum of $(t-s)^2$ in the interval Δ . By the mean-value theorem for Stieltjes integrals, and by (44),

$$b_\Delta (R_t' \xi, \xi) \leq \int_\Delta (t-s)^2 d(R_t' \xi, \xi) = 0.$$

On intervals containing s the minimum b_Δ is zero. On other intervals $b_\Delta > 0$, so that

$$\|\Delta \xi\|^2 = (\Delta \xi, \xi) = 0$$

for Δ not containing s . This means that $R_{s-\epsilon}' \xi$ is a constant vector for any $\epsilon > 0$. Or for $t < s$

$$(45) \quad R_t' \xi = \text{constant vector} \equiv \xi^*.$$

Now $R_t' \rightarrow \xi^*$ as $t \rightarrow -\infty$, the last equation gives

$$\xi^* = \lim_{t \rightarrow -\infty} R_t' \xi = 0.$$

Consequently (45) becomes

$$R_t' \xi = 0, \quad t < s.$$

From the left continuity of R_t' we infer that

$$0 = R_{s-0}' \xi = R_s' \xi.$$

Next, we observe that Ω is self-adjoint by Theorem 6. Moreover, the separation operators S_t of Ω are permutable with R_t' , since every R_t' permutes with Ω according to the same theorem. Also

we find without difficulty that

$$(46) \quad \Omega_t R_t' = \int_{-\infty}^t (r - t) dR_r' \leq Z,$$

as the integrand function is negative over $(-\infty, t]$. A glance at our proof of Theorem 4 reveals clearly that it still holds if we replace S_t' by S_t , S_t'' by R_t' , because the projection character of the two systems, their mutual permutability, and finally conditions (43), (46) enable us to repeat that proof word for word except the aforementioned change in notations. And the result is

$$R_t' = S_t.$$

Applying a similar proof to R_t'' we have at last

$$R_t' = S_t = R_t''.$$

Thus the unicity of representation is shown as stated in

THEOREM 10. If two left resolutions of the identity R_t' , R_t'' determine the same operator

$$\int_{-\infty}^{+\infty} t \, dR_t' = \Omega = \int_{-\infty}^{+\infty} t \, dR_t'',$$

then they are both equal, and coincide with the left continuous family of separations S_t of Ω .

Combining this with the previous results we can now state as an immediate corollary

THEOREM 11. In order that an operator Ω may be represented by a left resolution of the identity R_t in the form

$$\Omega = \int_{-\infty}^{+\infty} t \, dR_t,$$

it is necessary and sufficient that Ω be self-adjoint. The left separations of such an operator are equal to R_t .

Making use of Theorem 10 and Theorem 9 we obtain easily another unicity theorem:

THEOREM 12. For a given left resolution there exists one and only one self-adjoint operator Ω such that the left separations of Ω coincide with the resolution.

Because of these unicity considerations we shall discard the distinction of R_t and S_t , and, throughout the remainder of this paper, the notation E_t will be used for the family of separations or resolution of the identity associated with a self-adjoint operator.

5. The spectrum and its associated operators

Relative to a self-adjoint operator Ω the points on the real axis may be divided into three mutually exclusive sets:

(a) A point r interior to some interval on which E_t remains constant is said to belong to the resolvent set of Ω , $R(\Omega)$.

(b) If E_t is continuous¹ at s but not constant in any open interval containing s , then s lies in the continuous spectrum of Ω , $S_c(\Omega)$.

(c) We say that s belongs to the discontinuous spectrum of Ω , $S_d(\Omega)$, when E_t has a discontinuity at s .

The sum set of the continuous and discontinuous spectra is called the spectrum of Ω , $S(\Omega)$.

THEOREM 13. The spectrum of a self-adjoint operator Ω can never be empty. It is a closed set. We have, moreover, the following equalities:²

¹Owing to the projection character of E_t it is not necessary to distinguish the weak or the strong continuity--these two types are indeed equivalent here.

²The numbers $\bar{M}\Omega$, $\underline{M}\Omega$ have been defined in Chapter II.

$$\overline{ES}(\Omega) = \overline{M} \Omega \equiv \overline{M}, \quad \underline{ES}(\Omega) = \underline{M} \Omega \equiv \underline{M};$$

$$E_{\overline{M}+0} = I \text{ when } \overline{M} \text{ is finite,} \quad E_{\underline{M}-0} = Z \text{ when } \underline{M} \text{ is finite.}$$

In proof we note first that the theorem becomes trivial in a space having only zero vectors. Consider a space with some non-zero vectors. Suppose that the spectrum is empty. Then every real point belongs to the resolvent set, so that for every t there exists an open interval Δ containing t on which E_t is constant. Consequently E_t is continuous, and

$$d(E_t \mu, \xi)/dt = 0 \text{ everywhere.}$$

Thus, $E_t \mu$ is a constant, and as $t \rightarrow -\infty$ we find

$$E_t = Z \text{ for every } t,$$

which is impossible, since, by hypothesis, the space has some non-zero vectors. To establish the closure of the spectrum we must show that

$$E_{s+e} \neq E_{s-e}, \quad e > 0,$$

when

$$s_n \rightarrow s, \quad E_{s_n+e} \neq E_{s_n-e}, \quad e > 0.$$

Let e be a definite positive number different from zero. Then there is an s_n with $|s_n - s| < e/2$ so that

$$s - e < s_n \pm e/2 < s + e.$$

Put $E(t) \equiv (E_t \mu, \mu)$, $\|\mu\| \neq 0$. Now by the monotonicity of E_t ,

$$E(s_n + e/2) \leq E(s + e), \quad E(s - e) \leq E(s_n - e/2) \leq E(s_n + e/2).$$

This gives

$$\begin{aligned} E(s + e) - E(s - e) &\geq E(s_n + e/2) - E(s - e) \\ &\geq E(s_n + e/2) - E(s_n - e/2). \end{aligned}$$

Since E_t is not constant on any open interval containing s_n ,

$$E(s_n + e/2) \neq E(s_n - e/2).$$

Consequently,

$$E(s + e) \neq E(s - e).$$

As this argument applies to any e , the above inequality holds for every e and thus s belongs to the spectrum also.

Among the equalities we shall first prove

$$E_{\overline{M}+0} = I, \quad E_{\underline{M}-0} = Z.$$

From the definition of $\overline{M} \equiv \overline{M}\Omega$,

$$\int_{-\infty}^{+\infty} t \, d(E_t \{, \}) = (\Omega \{, \}) \leq \overline{M}, \quad \|\{, \}\| \leq 1.$$

So, when $\overline{M}$ is finite, $\overline{M}I - \Omega$ is a positive operator. Plainly then for $t > \overline{M}$ or $t = \overline{M} + e$,

$$(47) \quad (I - E_t)(\overline{M}I - \Omega) \geq Z,$$

because $I - E_t$ is positive. But

$$(48) \quad (I - E_t)(\overline{M}I - \Omega) = \int_t^{+\infty} (\overline{M} - s) dE_s.$$

As $\overline{M} - s$ is negative on $[t, +\infty)$, we have by the mean-value theorem

$$\int_t^{+\infty} (\overline{M} - s) dE_s \leq 0 \cdot \int_t^{+\infty} 1 \, dE_s \leq Z,$$

so that

$$\int_t^{+\infty} (s - \overline{M}) dE_s = Z,$$

in view of (47) and (48). Now $s - \bar{M} > e$ on $[t, +\infty)$, and applying the mean-value theorem again we find

$$Z = \int_t^{+\infty} (s - \bar{M}) dE_s \geq e \int_t^{+\infty} 1 dE_s \geq e(I - E_t).$$

It follows thus

$$(49) \quad E_t = I, \quad t > \bar{M},$$

since $e \neq 0$. Obviously, in the limit

$$E_{\bar{M}+0}^- = I.$$

Similar consideration shows that when $\underline{M}$ is finite

$$(50) \quad E_t = Z, \quad t < \underline{M}, \quad E_{\underline{M}-0} = Z.$$

Now we come to the equalities

$$\overline{BS}(\Omega) = \bar{M}\Omega, \quad \underline{BS}(\Omega) = \underline{M}\Omega.$$

Consider the case where $\bar{M}$ is finite. Then by (49) every $t > \bar{M}$ is interior to an interval on which E_s remains constant and hence belongs to the resolvent set. This means that no point in the spectrum can be greater than $\bar{M}$, or $\overline{BS}(\Omega) \leq \bar{M}$. It should be noted that for finite or infinite $\bar{M}$ we always have

$$(51) \quad \bar{B} = \overline{BS}(\Omega) \leq \bar{M}.$$

Next, suppose that $\bar{B}$ is finite. Then any point $t > \bar{B}$ lies in the resolvent set so that for every μ the function $(E_t \mu, \mu)$ is continuous and has its derivative zero everywhere in the open interval $(\bar{B}, +\infty)$. Clearly, E_t is constant there, and in fact $E_t = I$ for $t > \bar{B}$. Then the integral representation of Ω reduces to

$$\Omega = \int_{-\infty}^{\bar{B}+e} t dE_t, \quad e > 0.$$

For ξ in the independent range of Ω we have

$$(\Omega \xi, \xi) \leq (\overline{B} + e) \cdot \int_{-\infty}^{\overline{B}+e} 1 \, d(E_t \xi, \xi) \leq (\overline{B} + e) \cdot (E_{\overline{B}+e} \xi, \xi).$$

As e approaches zero,

$$(\Omega \xi, \xi) \leq \overline{B}(E_{\overline{B}} \xi, \xi) \leq \overline{B}(\xi, \xi).$$

Thus

$$(\Omega \xi, \xi) \leq \overline{B}, \quad \|\xi\| \leq 1,$$

which gives

$$(52) \quad \mathbb{M} \Omega \leq \overline{B},$$

whether $\overline{B}$ is finite or not. Combining the results in (51) and (52) we obtain the equality

$$\overline{B}S(\Omega) = \mathbb{M} \Omega.$$

The other equality follows from a similar argument.

We may note that the definitions of the resolvent set, continuous and discontinuous spectra are analytical. However, geometric characterizations can be given and in quite elementary form too.

Let us first consider a point r in the resolvent set $R(\Omega)$. Then r lies in an open interval $\Delta \equiv (t_1, t_2)$ where $E_{t_1} = E_{t_2}$.

If we now define

$$\begin{aligned} (53) \quad (\Omega - rI)^* &\equiv \int_{-\infty}^{+\infty} (1/(t - r)) dE_t \\ &\equiv \left(\int_{e=0}^{\frac{1}{S}} \int_{-\infty}^{r-e} + \int_{e=0}^{\frac{1}{S}} \int_{r+e}^{+\infty} \right) (1/(t - r)) dE_t, \end{aligned}$$

the integrals certainly exist, for, as e decreases indefinitely so that $t_1 < r - e < r + e < t_2$, the integrals $\int_{-\infty}^{r-e}$, $\int_{r+e}^{+\infty}$ remain constant, since $\Delta E_t = Z$. But in the intervals $(-\infty, r-e]$,

$[r + e, +\infty)$ the function $1/(t - r)$ is bounded. In consequence,

$$\int_{-\infty}^{r-e} (1/(t - r))^2 d(E_t \mu, \mu), \quad \int_{r+e}^{+\infty} (1/(t + r))^2 d(E_t \mu, \mu)$$

are both bounded for any $\|\mu\| \leq 1$. This means of course that $(\Omega - rI)^*$ is bounded. When E_t is constant in $\Delta = (t_1, t_2)$, plainly

$$\Omega - rI = \left(\int_{-\infty}^{r-e} + \int_{r+e}^{+\infty} \right) (t - r) dE_t.$$

We can easily see that

$$(54) \quad (\Omega - rI)^*(\Omega - rI) = I \text{ as on } \mathcal{M}(\Omega),$$

owing to the orthogonality relations of E_t , noted in (18) and (20).

On the other hand, by an argument similar to that used in the proof of Theorem 6 (especially pp. 101 bottom - 103) we can readily show that the integrals

$$\int_{-\infty}^{r-e} t dE_t \zeta, \quad \int_{r+e}^{+\infty} t dE_t \eta$$

exist, where vectors ζ, η are defined thus:

$$\zeta \equiv \int_{-\infty}^{r-e} (1/(t - r)) dE_t \mu, \quad \eta \equiv \int_{r+e}^{+\infty} (1/(t - r)) dE_t \mu.$$

Consequently

$$\left(\int_{-\infty}^{r-e} + \int_{r+e}^{+\infty} \right) t dE_t \zeta = \int_{-\infty}^{r-e} t dE_t \zeta, \quad \left(\int_{-\infty}^{r-e} + \int_{r+e}^{+\infty} \right) t dE_t \eta$$

exist, since $(-\infty, r - e]$, $[r + e, +\infty)$ are disjoint intervals.

So, $\zeta + \eta$ lies in the independent range of Ω and $(\Omega - rI)^*$ is on $\mathcal{M}$ to $\mathcal{M}(\Omega)$. Then after some elementary calculations with the integrals we find

$$(55) \quad (\Omega - rI)(\Omega - rI)^* = I \text{ as on } \mathcal{M}.$$

This result with (54) identifies $(\Omega - rI)^*$ as the reciprocal of

Ω_r . Conversely, if Ω_r has a bounded reciprocal Γ_r for some real number r , then, as we now show¹ E_t is constant on a certain open interval containing r . Let the modulus of Γ_r be M . Define

$$\Delta_* \equiv [r - 1/2M, r + 1/2M].$$

Now $\Delta_*\mu$ lies in $\mathcal{M}(\Omega)$ for any μ so that $\Delta_*\mu$ is in $\mathcal{M}(\Omega)$. On account of their reciprocal relation, Ω_r , Γ_r are permutable and hence

$$\Delta_*\mu = \Gamma_r \Omega_r \Delta_*\mu = \Gamma_r \int_{-\infty}^{+\infty} (t - r) dE_t \Delta_*\mu = \Gamma_r \int_{\Delta_*} (t - r) dE_t \mu.$$

As Γ_r is bounded,

$$\begin{aligned} \|\Delta_*\mu\| &\leq M \left\| \int_{\Delta_*} (t - r) dE_t \mu \right\| \leq M \cdot \left\| \int_{\Delta_*} (1/2M) dE_t \mu \right\| \\ &\leq M \cdot (1/2M) \|\Delta_*\mu\| \leq (1/2) \|\Delta_*\mu\|, \end{aligned}$$

which is impossible unless $\|\Delta_*\mu\| = 0$. Thus $\Delta_*\mu = 0$ for any μ . Because of its monotonicity, E_t is constant on Δ_* .

In case s belongs to the discontinuous spectrum the projection operator

$$\nabla_s \equiv E_{s+0} - E_{s-0}$$

is not zero. By the usual mean-value theorem we derive the inequalities

$$(s - e)(E_{s+e} - E_{s-e}) \leq \int_{s-e}^{s+e} t dE_t \leq (s + e)(E_{s+e} - E_{s-e}),$$

which give at once

$$(56) \quad s \nabla_s \leq \Omega \nabla_s \leq s \nabla_s, \quad \Omega \nabla_s = s \nabla_s,$$

as we know that $E_{s+e} - E_{s-e}$, ∇_s are all on $\mathcal{M}$ to $\mathcal{M}_1$, using the

¹See Riesz's Szeged paper, pp. 53-4.

condition stated in Theorem 8, and

$$\int_{s-e}^{s+e} t \, dE_t = \Omega(E_{s+e} - E_{s-e}).$$

Now, ∇_s is not zero so that from (56) there exists a vector ξ different from zero with

$$(57) \quad \Omega \xi = s \xi.$$

However, (57) is also sufficient for s to be in the discontinuous spectrum. To show the presence of discontinuity of E_t at s we return to the operator Φ_t introduced in (8'). It can be readily verified that

$$\Gamma_w \xi = (1/(s - w)) \xi, \quad \Gamma_{\bar{w}} \xi = (1/(s - \bar{w})) \xi,$$

and hence

$$\Phi_t \xi = \Gamma_w \Gamma_{\bar{w}} \Omega_t \xi = \Gamma_w \Gamma_{\bar{w}} (s - t) \xi = ((s - t)/(s^2 + |w|^2)) \xi.$$

But

$$E_t(\Omega) \xi = E_0(\Phi_t) \xi = E_0((s - t)/(s^2 + |w|^2)) \xi$$

which is zero when $s - t \geq 0$ and is ξ when $s - t < 0$. In consequence

$$(58) \quad E_t \xi = 0, \quad t \leq s; \quad E_t \xi = \xi, \quad t > s.$$

These imply that

$$(59) \quad \nabla_s \xi = E_{s+0} \xi - E_{s-0} \xi = \xi,$$

which shows the discontinuity of E_t at s , for ξ is different from zero.

When $\Omega - sI$ has an unbounded reciprocal Γ_s , obviously s cannot belong to the resolvent set since Γ_s is unbounded. Nor can it lie in the discontinuous spectrum, for, assuming s to be

in the discontinuous spectrum we would have

$$\Omega_s \nabla_s = Z, \quad \Gamma_s \Omega_s \nabla_s = Z \neq \nabla_s,$$

contradictory to the reciprocal relation of Ω_s and Γ_s . Thus, s must be in the continuous spectrum of Ω . On the other hand, suppose that s is a point in the continuous spectrum. We introduce an operator

$$\begin{aligned} (\Omega - sI)*\zeta &\equiv \int_{-\infty}^{+\infty} (1/(t-s)) dE_t \zeta \\ (60) \quad &\equiv \left(\lim_{e=0} \int_{-\infty}^{s-e} + \lim_{e=0} \int_{s+e}^{+\infty} \right) (1/(t-s)) dE_t \zeta, \end{aligned}$$

whenever the last two limiting integrals exist. Clearly, $(\Omega - sI)*$ is linear and hermitian. Let π_1 be the independent range of $(\Omega - sI)*$. It can be readily seen that for any μ the vectors

$$E_{s-e}\mu, \quad (I - E_{s+e})\mu$$

both lie in π_1 , since $1/(t-s)$ is bounded in the intervals $(-\infty, s-e]$, $[s+e, +\infty)$, and $E_t E_{s-e}$, $E_t (I - E_{s+e})$ become constant for t exterior to the respective intervals. Further, by hypothesis, E_t is continuous at s , so that

$$\mu = E_s \mu + (I - E_s) \mu = \lim_{e=0} E_{s-e} \mu + \lim_{e=0} (I - E_{s+e}) \mu.$$

Thus every vector in $\mathcal{M}$ can be approximated by a sequence of vectors from π_1 , hence π_1 is dense on $\mathcal{M}$. Again, by arguments completely parallel to those used on pages 101-3 we show that $(\Omega - sI)* - wI$ has a bounded reciprocal on $\mathcal{M}$ to π_1 for any non-real w . The operator $(\Omega - sI)*$ is then self-adjoint. Now we are in a position to connect $\mathcal{M}_1 = \mathcal{M}(\Omega)$ and π_1 . In fact, it will be seen that $(\Omega - sI)*$ transforms vectors of π_1 into those of $\mathcal{M}_1$. Let γ be $(\Omega - sI)*\zeta$ with ζ in π_1 . To show

that γ belongs to $\mathcal{M}_1$ we divide the real axis into three parts, $(-\infty, s-1]$, $[s-1, s+1]$, $[s+1, +\infty)$, and consider only $(-\infty, s-1]$, $[s+1, +\infty)$, as t, t^2 are continuous on the entire axis and bounded on $[s-1, s+1]$. Now take a partition π of $(-\infty, s-1]$. Then for any Δ in π

$$\begin{aligned} \|t_\Delta \Delta \gamma - \int_\Delta (t/(t-s)) dE_t \xi\|^2 &= \left\| \int_\Delta ((t_\Delta - t)/(t-s)) dE_t \xi \right\|^2 \\ &= \int_\Delta |(t_\Delta - t)/(t-s)|^2 d(E_t \xi, \xi) \\ &\leq (N\pi)^2 \int_\Delta |1/(t-s)| d(E_t \xi, \xi) \\ &\leq (N\pi)^2 \|\Delta \gamma\|^2. \end{aligned}$$

Evidently we have

$$\sum_\Delta \left\| t_\Delta \Delta \gamma - \int_\Delta (t/(t-s)) dE_t \xi \right\|^2 \leq (N\pi)^2 \|\gamma\|^2,$$

which, on account of the orthogonal relations of E_t , implies that the infinite series

$$(61) \quad \sum_\Delta (t_\Delta \Delta \gamma - \int_\Delta (t/(t-s)) dE_t \xi)$$

converges strongly and that

$$(62) \quad \left\| \sum_\Delta (t_\Delta \Delta \gamma - \int_\Delta (t/(t-s)) dE_t \xi) \right\|^2 \leq (N\pi)^2 \|\gamma\|^2.$$

However, on $(-\infty, s-1]$ the function $t/(t-s)$ is bounded, since $1/(t-s)$ is bounded and

$$|t/(t-s)| \leq 1 + |s| |1/(t-s)|.$$

Hence the integral

$$\int_{-\infty}^{s-1} (t/(t-s)) dE_t \mu$$

exists for every μ of $\mathcal{M}$. From the simple equality

$$(63) \quad \sum_\Delta \int_\Delta (t/(t-s)) dE_t \xi = \int_{-\infty}^{s-1} (t/(t-s)) dE_t \xi,$$

we find that

$$\sum_{\Delta}^{\pi} t_{\Delta} \Delta \eta$$

converges strongly because of (61). Then (62) and (63) give

$$(64) \quad \left\| \sum_{\Delta}^{\pi} t_{\Delta} \Delta \eta - \int_{-\infty}^{s-1} (t/(t-s)) dE_t \zeta \right\|^2 \leq (N\pi)^2 \|\eta\|^2$$

Thus

$$(65) \quad \lim_{N\pi=0} \sum_{\Delta}^{\pi} t_{\Delta} \Delta \eta = \int_{-\infty}^{s-1} (t/(t-s)) dE_t \zeta,$$

and by definition of Stieltjes integral,

$$(66) \quad \int_{-\infty}^{s-1} t dE_t \eta$$

exists. Similar considerations establish also the existence of

$$(67) \quad \int_{s+1}^{+\infty} t dE_t \eta.$$

Combining these results we see that

$$(68) \quad \int_{-\infty}^{+\infty} t dE_t \eta$$

exists, since

$$(69) \quad \int_{s-1}^{s+1} t dE_t \eta$$

always has meaning. We conclude that $(\Omega - sI)*\zeta$ is in $\mathcal{M}_1$ for any ζ in $\mathcal{N}_1$. It remains to evaluate the integral (69). Owing to the continuity of E_t at s , we can write

$$(70) \quad \int_{s-1}^{s+1} t dE_t = \left(\lim_{e=0} \int_{s-1}^{s-e} + \lim_{e=0} \int_{s+e}^{s+1} \right) t dE_t.$$

But

$$\int_{s-1}^{s-e} t dE_t \int_{-\infty}^{+\infty} (1/(t-s)) dE_t \zeta = \int_{s-1}^{s-e} t dE_t \int_{s-1}^{s-e} \frac{1}{t-s} dE_t \zeta,$$

$e < 1,$

and the right member readily reduces to

$$\int_{s-1}^{s-e} (t/(t-s)) dE_t \zeta,$$

if we consider the difference

$$\begin{aligned} t_{\Delta} \Delta E_t \int_{s-1}^{s-e} (1/(t-s)) dE_t \zeta - \int_{\Delta} (t/(t-s)) dE_t \zeta \\ = \int_{\Delta} ((t_{\Delta} - t)/(t-s)) dE_t \zeta. \end{aligned}$$

Thus

$$\begin{aligned} \int_{s-1}^{s-e} t dE_t (\Omega - sI) * \zeta &= \int_{s-1}^{s-e} (t/(t-s)) dE_t \zeta \\ &= \int_{s-1}^{s-e} ((t-s)/(t-s)) dE_t \zeta + \\ &\quad \int_{s-1}^{s-e} (s/(t-s)) dE_t \zeta. \end{aligned}$$

Similarly,

$$\begin{aligned} \int_{s+e}^{s+1} t dE_t (\Omega - sI) * \zeta &= \int_{s+e}^{s+1} ((t-s)/(t-s)) dE_t \zeta + \\ &\quad \int_{s+e}^{s+1} (s/(t-s)) dE_t \zeta. \end{aligned}$$

Passing to the limit and adding, we find by (70)

$$\int_{s-1}^{s+1} t dE_t (\Omega - sI) * \zeta = \int_{s-1}^{s+1} 1 dE_t \zeta + s \int_{s-1}^{s+1} (1/(t-s)) dE_t \zeta.$$

Using the results in (65)-(68) with this equality we derive

$$\begin{aligned} \Omega (\Omega - sI) * \zeta &= \int_{-\infty}^{+\infty} t dE_t (\Omega - sI) * \zeta = \int_{-\infty}^{+\infty} 1 dE_t \zeta + \\ &\quad s \int_{-\infty}^{+\infty} (1/(t-s)) dE_t \zeta, \end{aligned}$$

which means $(\Omega - sI) *$ is the right reciprocal of $\Omega - sI$. Further, we wish to show that it is also the left reciprocal of $\Omega - sI$.

The first question to settle is whether $\Omega \pi_1$ lies in π_1 . As $1/(t-s)$ is bounded and continuous on $(-\infty, s-1]$ and on $[s+1, +\infty)$, the integrals

$$\int_{-\infty}^{s-1} (1/(t-s)) dE_t \mu, \quad \int_{s+1}^{+\infty} (1/(t-s)) dE_t \mu$$

exist for any μ of $\mathcal{M}$. So, we are really concerned with $(s-1, s+1)$. Denote $(\Omega - sI)^\dagger$ by φ . For an interval Δ of $(s+e, s+1)$ we have

$$\begin{aligned} & \left\| (1/(t_\Delta - s)) \Delta \varphi - \int_\Delta ((t-s)/(t-s)) dE_t \right\|^2 \\ &= \left\| \int_\Delta (1/(t_\Delta - s) - 1/(t-s))(t-s) dE_t \right\|^2 \\ &\leq e^2 \left\| \int_\Delta (t-s) dE_t \right\|^2 \leq e^2 \int_\Delta (t-s)^2 d(E_t^\dagger, \dagger) \end{aligned}$$

when the length of Δ is less than some d_e , since $1/(t-s)$ is bounded and continuous on $[s+e, s+1]$. If π is a partition of $(s+e, s+1)$ with $N\pi < d_e$,

$$\sum_{\Delta} \left\| (1/(t_\Delta - s)) \Delta \varphi - \int_\Delta 1 dE_t \right\|^2 \leq e^2 \int_{s+e}^{s+1} (t-s)^2 d(E_t^\dagger, \dagger).$$

Accordingly,

$$\left\| \sum_{\Delta} (1/(t_\Delta - s)) \Delta \varphi - \int_{s+e}^{s+1} 1 dE_t \right\|^2 \leq e^2 \int_{s+e}^{s+1} (t-s)^2 d(E_t^\dagger, \dagger),$$

and hence

$$\int_{s+e}^{s+1} (1/(t-s)) dE_t \varphi = \int_{s+e}^{s+1} 1 dE_t^\dagger = (E_{s+1} - E_{s+e})^\dagger.$$

Now, as $e \rightarrow 0$ the right member converges strongly to $(E_{s+1} - E_s)^\dagger$, due to the assumed continuity¹ of E_t at s . Consequently

$$(71) \quad \lim_{e=0} \int_{s+e}^{s+1} (1/(t-s)) dE_t \varphi = (E_{s+1} - E_s)^\dagger.$$

For the interval $[s-1, s-e]$ we readily deduce in a similar manner

$$(72) \quad \lim_{e=0} \int_{s-1}^{s-e} (1/(t-s)) dE_t \varphi = (E_s - E_{s-1})^\dagger.$$

¹If the continuity of E_t at s is absent then the integral limit still exists but has the value $(E_{s+1} - E_{s+0})^\dagger$, and because of this fact $(\Omega - sI) * (\Omega - sI)^\dagger$ will then be different from $\dagger$. Cf. equation (74) below.

Hence

$$\int_{s-1}^{s+1} (1/(t-s)) dE_t \varphi$$

exists, and $(\Omega - sI)\xi$, $\Omega\xi$ belong to $\mathcal{K}_1$ for any ξ of $\mathcal{M}_1$.

Without difficulty we can verify that

$$(73) \quad \int_{-\infty}^{s-1} (1/(t-s)) dE_t \varphi = E_{s-1} \xi, \quad \int_{s+1}^{+\infty} (1/(t-s)) dE_t \varphi = (I - E_{s+1}) \xi.$$

In view of (71), (72), we find through mere addition

$$(74) \quad \int_{-\infty}^{+\infty} (1/(t-s)) dE_t \varphi = E_{s-1} \xi + (E_s - E_{s-1}) \xi + (E_{s+1} - E_s) \xi + (I - E_{s+1}) \xi.$$

Thus $(\Omega - sI)^* \varphi = \xi$ and $(\Omega - sI)^*$ is therefore the left reciprocal of $\Omega - sI$. So far, we have proved that if s lies in the continuous spectrum of Ω then there exists a self-adjoint operator $(\Omega - sI)^*$ on $\mathcal{K}_1$ such that

$$\Omega \mathcal{K}_1 = \mathcal{K}_1, \quad (\Omega - sI)^* \mathcal{K}_1 = \mathcal{K}_1,$$

and $(\Omega - sI)^*$ is the reciprocal of $\Omega - sI$. Moreover, $(\Omega - sI)^*$ must be unbounded, for, if it were bounded s would belong to the resolvent set, but s is assumed to be a point of the continuous spectrum. From this unboundedness we then conclude that $\mathcal{K}_1$ is necessarily a proper subspace of $\mathcal{M}$, because a hermitian operator defined on the entire complete space is always bounded.¹

Thus we have established the next theorem.²

THEOREM 14. The points in the continuum of real numbers are divided into three disjoint sets which may be characterized as follows:

¹See p. 36.

²Cf. Stone's first and second notes.

(a) A point r lies in the resolvent set of a self-adjoint operator, if and only if $\Omega - rI$ has a bounded reciprocal Γ_r . In this case,

$$\Gamma_r = \int_{-\infty}^{+\infty} (1/(t - r)) dE_t.$$

(b) For s to be a point of the continuous spectrum it is necessary and sufficient that there exists a unique unbounded reciprocal Γ_s of $\Omega - sI$. Under such a condition Γ_s is self-adjoint and operates only on a proper linear subspace $\mathcal{M}_1$ dense in $\mathcal{M}$ with

$$\pi_1 = \Omega \pi_1, \pi_1 = \Gamma_s \pi_1, \pi_1 \equiv \mathcal{M}(\Omega),$$

$$\Gamma_s = \int_{-\infty}^{+\infty} (1/(t - s)) dE_t \text{ as on } \mathcal{M}_1.$$

(c) When, and only when, s belongs to the discontinuous spectrum, there is at least one vector ξ of $\mathcal{M}_1$ for which $(\Omega - sI)\xi = 0$ while $\|\xi\| > 0$.

As a preparatory theorem of special use we state and prove

THEOREM 15. A necessary and sufficient condition for the existence of a bounded reciprocal to a self-adjoint operator Φ is that

$$(75) \quad \|\Phi\xi\| \geq e \|\xi\|$$

for ξ in $\mathcal{M}(\Phi)$, e being a positive constant different from zero. Moreover,

$$(76) \quad \begin{aligned} R(\Phi^{-1}) &= 1/R(\Phi) - [0] + ([0] \text{ when } \Phi \text{ is bounded}), \\ S(\Phi^{-1}) &= 1/S(\Phi) + ([0] \text{ when } \Phi \text{ is unbounded}). \end{aligned}$$

The theorem is trivial for a space consisting of one zero vector only. To show the necessity of the condition let us assume

the finite modulus of Φ^{-1} to be N which is of course greater than zero. Then, for ξ in $\mathcal{M}(\Phi)$ we have

$$\Phi\xi = \mu, \quad \Phi^{-1}\mu = \xi, \quad \|\xi\| = \|\Phi^{-1}\mu\| \leq N\|\mu\| \leq N\|\Omega\xi\|.$$

The last set of inequalities gives immediately

$$(1/N)\|\xi\| \leq \|\Omega\xi\|,$$

so that on putting $\epsilon \equiv 1/N$ we have the desired test. For the sufficiency we note first that $\mathcal{N}_1 \equiv \Phi\mathcal{M}_1$ is linear. It is also dense on $\mathcal{M}$. Suppose that it is not dense. There is then a non-zero vector ξ orthogonal to $\mathcal{N}_1$, hence

$$0 = (\mathcal{N}_1, \xi) = (\Phi\mathcal{M}_1, \xi) = (\mathcal{M}_1, 0).$$

This, by virtue of the self-adjoint property of Φ , means that ξ belongs to $\mathcal{M}_1$ and $\Phi\xi = 0$, which shows $\|\Phi\xi\| = 0$, violating the initial condition (75). So, $\mathcal{N}_1$ must be dense. Further, we assert that $\mathcal{N}_1 = \mathcal{M}$. To every μ there exists a sequence $\{\mu_m\}$ of vectors from $\mathcal{N}_1$, converging strongly to μ . Consequently there is a corresponding sequence $\{\xi_m\}$ such that $\Phi\xi_m = \mu_m$.

Hence

$$(78) \quad \lim_{m \rightarrow \infty} \Phi\xi_m = \mu$$

But from the obvious relations

$$(79) \quad \epsilon \|\xi_m\| \leq \|\Phi\xi_m\| = \|\mu_m\|$$

we derive

$$\|\xi_m - \xi_n\| \leq \frac{1}{\epsilon} \|\mu_m - \mu_n\| \rightarrow 0$$

As Φ is a closed operator by Theorem 1, there exists a vector ξ with

$$(80) \quad \Phi \xi = \lim_{m \rightarrow \infty} \Phi \xi_m = \mu,$$

because of (78) and the strong convergence of $\{\xi_m\}$. Define the sought reciprocal by

$$\Phi^{-1}\mu \equiv \xi$$

for every μ of $\mathcal{M}$. With the aid of (79), (80) we see that

$$e\|\xi\| \leq \|\mu\|,$$

and hence the modulus of Φ^{-1} does not exceed $1/e$.

To establish the equalities in the second part of the theorem we consider first a point $r \neq 0$ in the resolvent set of Φ , so that $\Omega - rI$ has a bounded reciprocal by (a) of Theorem 14. It is to be seen that $\Phi^{-1} - (1/r)I$ also has a bounded reciprocal equal to $r\Phi(rI - \Phi)^{-1}$. By direct computation,

$$(81) \quad (\Phi^{-1} - I/r)[r\Phi(rI - \Phi)^{-1}] = (rI - \Phi)(rI - \Phi)^{-1} = I \text{ on } \mathcal{M}.$$

On account of the permutability of $\Phi - rI$, $(\Phi - rI)^{-1}$, Φ permutes with $(rI - \Phi)^{-1}$ on $\mathcal{M}(\Phi)$. As Φ^{-1} and $(rI - \Phi)^{-1}$ are on $\mathcal{M}$ to $\mathcal{M}(\Phi)$, Φ^{-1} permutes with $\Phi(rI - \Phi)^{-1}$. Consequently,

$$r\Phi(rI - \Phi)^{-1}, \quad \Phi^{-1} - I/r,$$

are permutable, and hence by (81)

$$[r\Phi(rI - \Phi)^{-1}](\Phi^{-1} - I/r) = I \text{ on } \mathcal{M}.$$

Further, the fact that the reciprocal $r\Phi(rI - \Phi)^{-1}$ operates on the entire space shows its boundedness. This in turn implies that $1/r$ belongs to the resolvent set of Φ^{-1} . On the other hand, suppose that $\lambda \neq 0$ is a point in the resolvent set of Φ^{-1} . Then $(\Phi^{-1} - \lambda I)^{-1}$ is bounded, and

$$(82) \quad (I/\chi - \Phi)[\chi\Phi^{-1}(\Phi^{-1} - \chi I)^{-1}] = I \text{ on } \mathcal{M}.$$

From the permutability of Φ^{-1} with $(\Phi^{-1} - \chi I)^{-1}$ we infer that Φ permutes with $\Phi^{-1}(\Phi^{-1} - \chi I)^{-1}$. Because of (82) and the last permutability,

$$\chi\Phi^{-1}(\Phi^{-1} - \chi I)^{-1}(I/\chi - \Phi) = I \text{ on } \mathcal{M}(\Phi).$$

Thus $I/\chi - \Phi$ has the reciprocal $\chi\Phi^{-1}(\Phi^{-1} - \chi I)^{-1}$ which is indeed bounded. Evidently $1/\chi$ lies in the resolvent set of Φ . By virtue of Theorem 14, 0 belongs to $R(\Phi^{-1})$ when $\Phi = (\Phi^{-1})^{-1}$ is bounded. So, we have the equality in (76). For (77) we observe that 0 lies in the spectrum of Φ^{-1} if Φ is unbounded by (b) of Theorem 14. Arguing indirectly with the aid of (76) we readily deduce (77), due to the mutual complementary character of the spectrum and the resolvent set.

The spectrum of an operator can also be characterized by another criterion¹ which closely resembles that for the discontinuous spectrum.

THEOREM 16. A point s on the real axis belongs to the spectrum of a self-adjoint operator Ω in a non-zero space, if and only if there exists a weakly convergent sequence of vectors $\{\xi_n\}$ such that

$$(83) \quad \|\xi_n\| = 1, \quad \lim_{n \rightarrow \infty} (\Omega - sI)\xi_n = 0.$$

In case s lies in the continuous spectrum, any sequence $\{\xi_n\}$ fulfilling the conditions in (83) must converge weakly to zero.²

¹Cf. Weyl, Palermo Rend. 27 (1909), p. 378; Carleman, p. 112.

²For a point in the spectrum of a hermitian operator in n -dimensional euclidean space this is impossible.

From Theorem 15, we see that $\Omega - sI$ fails to have a bounded reciprocal when the greatest lower bound of $\|(\Omega - sI)\xi\|$ for $\|\xi\| = 1$ is zero. If (83) holds, $\Omega - sI$ indeed can have no bounded reciprocal and s is then in the spectrum. However, suppose that s belongs to the spectrum. Then $\Omega - sI$ has no bounded reciprocal and so the greatest lower bound of $\|(\Omega - sI)\xi\|$ for $\|\xi\| = 1$ is certainly zero, in view of the previous theorem. Accordingly there exists a sequence $\{\xi_n\}$ that satisfies (83). But the space is weakly compact (see Chapter I) so that we can find an infinite subsequence converging weakly. Now, consider the particular case when s is a point of the continuous spectrum. Then $(\Omega - sI)^{-1}$ exists, and

$$(\xi, \xi_n) = (\xi, \Gamma_\Omega \Omega_\Omega \xi_n) = (\Gamma_\Omega \xi, \Omega_\Omega \xi_n) \rightarrow 0$$

for every ξ in the independent range of $(\Omega - sI)^{-1}$ which is dense on $\mathcal{M}$. It follows that for any μ of $\mathcal{M}$

$$(\mu, \xi_n) \rightarrow 0.$$

As $\|\xi_n\| = 1$ is uniformly bounded, this means that $\{\xi_n\}$ converges weakly to the zero vector.

We note that the spectral content of the reciprocals $\Gamma_w \equiv (\Omega - wI)^{-1}$ can be described in terms of that of Ω .

THEOREM 17. The resolvent set and the spectra of Γ_w are given as follows:

$$(84) \quad R(\Gamma_w) = 1/R'(\Omega) - w + ([0] \text{ when } \Omega \text{ is bounded}),$$

where $R'(\Omega)$ stands for $R(\Omega) - w$ or $R(\Omega)$ according as w belongs to $R(\Omega)$ or not;

$$(85) \quad S_c(\Gamma_w) = 1/(S_c'(\Omega) - w) + ([0] \text{ when } \Omega \text{ is unbounded}),$$

where $S_c'(\Omega)$ has similar meaning as $R'(\Omega)$;

$$(86) \quad S_d(\Gamma_w) = 1/(S_d(\Omega) - w).$$

If $\lambda \neq w$ is in the resolvent set of Ω then $\lambda - w$ is in the resolvent set of $\Omega - wI$ and hence $1/(\lambda - w)$ belongs to the resolvent set of Γ_w by Theorem 15. Conversely, when $m \neq 0$ lies in the resolvent set of Γ_w , then $1/m$ cannot be in the spectrum of $\Omega - wI$, for if it were, $1/(1/m)$ would be a point in the spectrum of Γ_w , again because of Theorem 15. So, $1/m$ is in the resolvent set of $\Omega - wI$ and $1/m + w$ is then a resolvent point of $\Omega = (\Omega - wI) + wI$. According to Theorem 14, 0 belongs to the resolvent set of Γ_w if and only if $\Omega - wI$ and hence Ω is bounded. On account of the mutual exclusiveness of the spectrum and the resolvent set we readily conclude by indirect argument that

$$(87) \quad S(\Gamma_w) = 1/(S'(\Omega) - w) + ([0] \text{ when } \Omega \text{ is unbounded}),$$

$S'(\Omega)$ being $S(\Omega) - [w]$ when w is in the spectrum of Ω . Suppose now that there exists a non-zero vector ξ with $(\Omega - \lambda I)\xi = 0$. Then

$$(\Omega - wI)\xi = (\lambda - w)\xi, \quad \xi = \Gamma_w(\Omega - wI)\xi = (\lambda - w)\Gamma_w\xi,$$

so that $1/(\lambda - w)$ is a point in the discontinuous spectrum of Γ_w . On the other hand, for $\xi \neq 0$, $\Gamma_w\xi = m\xi$ (hence $m \neq 0$), we have

$$m\Omega\xi = \Omega\Gamma_w\xi = \xi + w\Gamma_w\xi = \xi + wm\xi.$$

Thus $1/m + w$ lies in the discontinuous spectrum of Ω , and (86) is proved. To show the equality in (85) we need only apply (87) in conjunction with (86), noting however that, in virtue of Theorem 14, if 0 lies in the spectrum of Γ_w at all it must be a point of the continuous spectrum and this is the case when and

only when Ω is unbounded.

In the remaining pages of this section we shall be concerned with decompositions of a resolution of the identity.

Let $\mathcal{M}_d$ be the extension of all characteristic vectors of Ω such that there is an r with $\Omega \xi = r \xi$. Introduce the projection operator

$$D \equiv P_{\mathcal{M}_d},$$

and $D \equiv Z$ when Ω has no characteristic vectors. Now by (59),

$\nabla_s = P_{\mathcal{K}_s}$, where $\mathcal{K}_s$ is the set of all vectors ξ_s such that $\Omega \xi_s = s \xi_s$. Evidently $\mathcal{K}_s$ is linear, closed, and is the corresponding space of the projection operator $P_{\mathcal{K}_s}$. But we verify without difficulty that

$$\nabla_r \nabla_s = Z, \quad r \neq s.$$

According to a theorem on monotone sequence of projection operators in Chapter II then

$$\sum_{r < t} \nabla_r$$

exists as a strong limit and is the projection of the sum-space of the spaces $\nabla_r \mathcal{M} = \mathcal{K}_r$ with $r < t$. (As the non-vanishing terms may be uncountably many the limit is taken in the sense of general analysis.) Consequently

$$D \equiv P_{\mathcal{M}_d} = \sum_r \nabla_r,$$

and hence D permutes with E_t . On the other hand,

$$\begin{aligned} E_t \nabla_s &= Z, & t \leq s, \\ E_t \nabla_s &= \nabla_s, & t > s, \end{aligned}$$

from the very definition of ∇_s and the orthogonal properties of

E_t . i. we set

$$D_t \equiv DE_t,$$

we see that

$$D_t = \left(\sum_r \nabla_r \right) E_t = \sum_{r < t} \nabla_r,$$

from which follows

$$D_t \nabla_s = Z, \quad t \leq s,$$

$$D_t \nabla_s = \nabla_s, \quad t > s.$$

It should be noted that for each μ the sum

$$\sum_{r < t} \nabla_r \mu$$

has only a countable number of non-vanishing terms, because of the character of the corresponding numerical series of positive terms:

$$\sum_{r < t} (\nabla_r \mu, \mu) = \sum_{r < t} \|\nabla_r \mu\|^2.$$

Since the non-vanishing terms $(\nabla_r \mu, \mu)$ represent the jumps of the function of limited variation $(E_t \mu, \mu)$, $(D_t \mu, \mu)$ is the jump function¹ of $(E_t \mu, \mu)$. Polarizing this result we have $(D_t \mu, \xi)$ as the jump function of $(E_t \mu, \xi)$. In the present case D_t is however left continuous in the strong sense, as

$$\lim_{e \rightarrow 0} D_{t-e} = \lim_{e \rightarrow 0} DE_{t-e} = DE_t = D_t, \quad e > 0.$$

Define

$$C_t \equiv E_t - D_t.$$

Then C_t, D_t are orthogonal. From the obvious relations

¹See, for example, Hahn, Theorie der reellen Funktionen, vol. 1, pp. 505 ff.

$$\Delta D_t = \Delta (DE_t) = D\Delta E_t \rightarrow D\nabla_s = \nabla_s,$$

$$\Delta C_t = \Delta E_t - \Delta D_t,$$

we find

$$D_{t+0} - D_{t-0} = E_{t+0} - E_{t-0},$$

$$C_{t+0} - C_{t-0} = Z,$$

and C_t is strongly continuous in t . We notice also

$$(\Delta D_t)^2 = \Delta D_t, \quad (\Delta C_t)^2 = \Delta C_t,$$

$$D_t \rightarrow Z, \quad C_t \rightarrow Z, \quad t \rightarrow -\infty,$$

after a few simple calculations. The equalities

$$E_t D_t = D_t, \quad E_t C_t = C_t$$

imply that D_t, C_t are all on $\mathcal{M}_1$ to $\mathcal{M}_1$. We need hardly mention that D_t, C_t both permute with Ω on $\mathcal{M}_1$. These considerations lead to

THEOREM 18. Let $\mathcal{M}_d$ be the smallest linear complete space containing all characteristic vectors of the self-adjoint operator, Ω , and D the projection operator of $\mathcal{M}_d$. Then the mutually permutable operators

$$\nabla_s \equiv E_{r+0} - E_{s-0}, \quad D_t \equiv DE_t, \quad C_t \equiv E_t - D_t$$

have the following properties and relations:¹

(88) ∇_s is not identically zero if and only if s belongs to the discontinuous spectrum of Ω ;

$$(89) \quad \begin{aligned} \nabla_r \nabla_s &= Z, \quad r \neq s; \quad \Omega \nabla_s = s \nabla_s; \quad E_r \nabla_s = \nabla_s, \quad r > s, \\ E_r \nabla_s &= Z, \quad r \leq s; \quad D_r D_s = D_r, \quad r \leq s; \quad E_r D_t = D_t, \quad r > t, \end{aligned}$$

¹Cf. Carleman, pp. 88-91.

$$(89) E_r D_t = D_r, r \leq t; D_t \rightarrow Z, t \rightarrow -\infty; D_t = P_{\pi_t^*} = \sum_{r < t} \nabla_r,$$

where π_t^* is the sum-space of all characteristic vectors with characteristic values less than t , and the strongly convergent sum is taken in the sense of general analysis, as there may be uncountably many operators ∇_s that do not vanish; D_t is left continuous;

$$(90) C_t \text{ is continuous in } t;$$

$$C_r C_s = C_r, r \leq s; \nabla_r C_s = Z; D_r C_s = Z;$$

$$E_r C_t = C_t, r > t, E_r C_t = C_r, r \leq t; C_t \rightarrow Z, t \rightarrow -\infty.$$

Moreover, the operators D_t, C_t permute with Ω as on $\pi(\Omega)$.

$(D_t \mu, \xi)$ is the discontinuous jump function, while $(C_t \mu, \xi)$ is the continuous part of $(E_t \mu, \xi)$ as a function of limited variation.

Without difficulty we can obtain resolutions of the identity relative to the operators

$$\Omega_d \equiv \int_{-\infty}^{+\infty} t \, dD_t, \quad \Omega_c \equiv \int_{-\infty}^{+\infty} t \, dC_t,$$

or the discontinuous and continuous parts of Ω . Put

$$C \equiv I - D,$$

$$E_{dt} \equiv D_t, t \leq 0; E_{dt} \equiv D_t + C, t > 0;$$

$$E_{ct} \equiv C_t, t \leq 0; E_{ct} \equiv C_t + D, t > 0.$$

Then we verify readily that the two families E_{dt}, E_{ct} are resolutions of the identity if we recall that

$$E_{dt} \rightarrow D + (I - D) = I, E_{ct} = (E_t - D_t) + D \rightarrow (I - D) + D = I,$$

as $t \rightarrow +\infty$. Further,

$$\Omega_d = \int_{-\infty}^{+\infty} t \, dE_{dt}.$$

For, using the mean-value theorem,

$$\Delta_e \equiv (-e, +e), \quad e > 0,$$

$$-e \Delta_e D_t \leq \int_{-e}^{+e} t \, dD_t \leq e \Delta_e D_t, \quad -e \Delta_e E_{dt} \leq \int_{-e}^{+e} t \, dE_{dt} \leq e \Delta_e E_{dt}.$$

As

$$\int_{-\infty}^{-e} t \, dD_t = \int_{-\infty}^{-e} t \, dE_{dt}, \quad \int_{+e}^{+\infty} t \, dD_t = \int_{+e}^{+\infty} t \, dE_{dt},$$

$$\Omega_d = \left(\int_{-\infty}^{-e} + \int_{-e}^{+e} + \int_{+e}^{+\infty} \right) t \, dD_t,$$

we find

$$\Omega_d - \left(\int_{-\infty}^{-e} + \int_{+e}^{+\infty} \right) t \, dE_{dt} = \int_{-e}^{+e} t \, dD_t \rightarrow 0, \quad e \rightarrow 0.$$

It follows that

$$\Omega_d = \left(\lim_{e \rightarrow 0} \int_{-\infty}^{-e} + \lim_{e \rightarrow 0} \int_{+e}^{+\infty} \right) t \, dE_{dt} = \int_{-\infty}^{+\infty} t \, dE_{dt}.$$

So, by the unicity of representation of a self-adjoint operator (Theorem 10), E_{dt} is the resolution of the identity relative to Ω_d . Similarly we can easily show that E_{ct} is the resolution relative to Ω_c . Thus we have

THEOREM 19. The continuous and discontinuous parts of the self-adjoint Ω :

$$\Omega_c \equiv \int_{-\infty}^{+\infty} t \, dC_t, \quad \Omega_d \equiv \int_{-\infty}^{+\infty} t \, dD_t,$$

have respectively the following resolutions of the identity:

$$E_{ct} = C_t, \quad t \leq 0; \quad E_{ct} = C_t + D, \quad t > 0;$$

$$E_{dt} = D_t, \quad t \leq 0; \quad E_{dt} = D_t + (I - D), \quad t > 0.$$

It is customary to consider the continuous and discontinuous parts of E_t as in Theorem 18. However, in real function theory a function of limited variation consists of three unique parts:

a discontinuous function having denumerably many jumps, a barely continuous function with the derivative vanishing almost everywhere and an absolutely continuous function equal to the indefinite integral of its derivative. May we then expect the analogy to go deeper? This is indeed the case.

THEOREM 20. Any resolution of the identity $\{E_t\}$ can be uniquely expressed as the sum of three mutually orthogonal families of projection operators:

$$E_t = D_t + B_t + A_t,$$

where $(D_t\mu, \xi)$ is the discontinuous jump part, $(B_t\mu, \xi)$ the barely continuous singular part, $(A_t\mu, \xi)$ the absolutely continuous regular part of $(E_t\mu, \xi)$. In fact, D_t, B_t, A_t vanish at $-\infty$, and are respectively weakly discontinuous, strongly continuous, strongly absolutely continuous in the operatorial sense; $(B_t\mu, \xi)$ has a derivative almost everywhere zero, $(A_t\mu, \xi)$ equals the indefinite integral of its derivative. Further,

$$D_t = \sum_{r < t} (E_{r+0} - E_{r-0}),$$

$$(A_t\mu, \xi) = \int_{-\infty}^t (d/dx)((E_x - D_x)\mu, \xi) dx.$$

In view of Theorem 18 we have only to further separate C_t into two parts:

$$C_t = B_t + A_t.$$

The problem then reduces to the finding of an absolutely continuous function A_t such that $((C_t - A_t)\mu, \xi)$ has an almost vanishing derivative. From the fact that $(\mu, C_x \xi)$ is a function of limited variation over the entire infinite interval we see that it has a finite derivative almost everywhere on $(-\infty, +\infty)$. The monotone

increasing property of C_x shows also that the derivative is positive. According to the standard results in function theory this derivative is moreover Lebesgue integrable over any finite interval, and

$$0 \leq \int_{\Delta} (d/dx)(\mu, C_x \mu) dx \leq \Delta(\mu, C_x \mu).$$

As C_x is of limited variation, this implies that the derivative of $(\mu, C_x \mu)$ is integrable over the entire infinite interval. With the aid of polarization we conclude that the derivative of $(\mu, C_x \xi)$ is again integrable over $(-\infty, +\infty)$, and

$$(91) \quad \int_{-\infty}^{+\infty} |(d/dx)(\mu, C_x \xi)| dx \leq \|\mu\| \cdot \|\xi\|.$$

Now introduce the linear functional $A_t(\mu, \xi)$ by

$$A_t(\mu, \xi) \equiv \int_{-\infty}^t (d/dx)(\mu, C_x \xi) dx.$$

Clearly it is positive hermitian, and, on account of (91), bounded also. These in turn point to the fact that the functional gives rise, via the representation theorem in Chapter I, to an operator A_t :

$$(92) \quad (\mu, A_t \xi) = A_t(\mu, \xi) = \int_{-\infty}^t (d/dx)(\mu, C_x \xi) dx,$$

so that A_t is a bounded positive hermitian operator. The integral on the right hand side shows the absolute continuity of A_t in the weak sense.

To exhibit the projection character of A_t we proceed thus: First, by (92)

$$(93) \quad \begin{aligned} (A_s \mu, A_r \xi) &= \int_{-\infty}^r (d/dx)(A_s \mu, C_x \xi) dx \\ &= \int_{-\infty}^r \left[(d/dx) \int_{-\infty}^s (d/dy)(C_y \mu, C_x \xi) dy \right] dx. \end{aligned}$$

Next, consider $r \leq s$, and an interval $\Delta \equiv (x_1, x_2)$ in $(-\infty, r)$.

Then

$$\begin{aligned}
 (\Delta/\Delta x) \int_{-\infty}^s (d/dy)(C_y \mu, C_x \xi) dy &= (1/\Delta x) \int_{-\infty}^s (d/dy)(C_y \mu, \Delta C_x \xi) dy \\
 &= (1/\Delta x) \int_{\Delta} (d/dy)(C_y \mu, \Delta C_x \xi) dy \\
 &= (1/\Delta x) \left(\int_{-\infty}^{x_2} - \int_{-\infty}^{x_1} \right) (d/dy)(C_y \mu, C_{x_2} \xi) dy \\
 &\quad - (1/\Delta x) \left(\int_{-\infty}^{x_2} - \int_{-\infty}^{x_1} \right) (d/dy)(C_y \mu, C_{x_1} \xi) dy \\
 &= (1/\Delta x) \left[\int_{-\infty}^{x_2} (d/dy)(C_y \mu, C_{x_2} \xi) dy - \int_{-\infty}^{x_1} (d/dy)(C_y \mu, C_{x_1} \xi) dy \right] \\
 &\quad - (1/\Delta x) \left[\int_{-\infty}^{x_1} (d/dy)(C_y \mu, C_{x_1} \xi) dy - \int_{-\infty}^{x_1} (d/dy)(C_y \mu, C_{x_1} \xi) dy \right],
 \end{aligned}$$

due to the orthogonal relations of C_t arising out of (90). In consequence,

$$\begin{aligned}
 (\Delta/\Delta x) \int_{-\infty}^s (d/dy)(C_y \mu, C_x \xi) dy &= (\Delta/\Delta x) \int_{-\infty}^x (d/dy)(C_y \mu, C_x \xi) dy \\
 &= (\Delta/\Delta x) \int_{-\infty}^x (d/dy)(C_y \mu, \xi) dy,
 \end{aligned}$$

which gives, in the limit,

$$\begin{aligned}
 (d/dx) \int_{-\infty}^s (d/dy)(C_y \mu, C_x \xi) dy &= (d/dx) \int_{-\infty}^x (d/dy)(C_y \mu, \xi) dy \\
 &= (d/dx)(C_x \mu, \xi) = (d/dx)(\mu, C_x \xi), \quad x \leq r \leq s.
 \end{aligned}$$

Integrating both sides,

$$\begin{aligned}
 \int_{-\infty}^r \left[(d/dx) \int_{-\infty}^s (d/dy)(C_y \mu, C_x \xi) dy \right] dx &= \int_{-\infty}^r (d/dx)(\mu, C_x \xi) dx \\
 &= (\mu, A_r \xi).
 \end{aligned}$$

By comparison with (93),

$$(A_s \mu, A_r \xi) = (\mu, A_r \xi), \quad r \leq s.$$

This means precisely that

$$A_s A_r = A_r = A_r A_s, \quad r \leq s.$$

Thus $A_t^2 = A_t$, hence A_t is a projection operator.

It is then a simple matter to deduce the strong absolute continuity of A_t . By virtue of the projection character of A_t we have

$$\|\Delta A_t \mu\|^2 = \Delta(A_t \mu, \mu) \rightarrow 0, \quad \Delta \rightarrow 0,$$

since A_t is absolutely continuous in the weak sense.

By the very definition in (92), $(A_t \mu, \xi)$ coincides with the unique absolutely continuous part of $(C_t \mu, \xi)$. Consequently, if we put

$$B_t \equiv C_t - A_t,$$

$(B_t \mu, \xi)$ is the barely continuous function whose derivative vanishes almost everywhere according to real function theory. Of course, B_t is a projection operator, since C_t, A_t are such. To see the mutual orthogonality of these families, consider, for example, A_s and D_t . Since $A_s \leq C_s$,

$$A_s D_t = A_s C_s D_t = A_s Z.$$

Other orthogonalities follow from similar arguments.

Finally we may remark that the resolutions of the identity relative to

$$\Omega_b \equiv \int_{-\infty}^{+\infty} t \, dB_t, \quad \Omega_a \equiv \int_{-\infty}^{+\infty} t \, dA_t$$

can be simply expressed in terms of B_t, A_t respectively, as in Theorem 19.

CHAPTER V
CHARACTERISTIC EXPANSIONS

1. Preparatory theorems

For an arbitrary real-valued function $G(t)$ let us introduce the truncated function $G_N(t)$:

$$\begin{aligned} G_N(t) &\equiv G(t) \text{ when } -N < G(t) < N \\ &\equiv N \quad \text{when } G(t) \geq N \\ &\equiv -N \quad \text{when } G(t) \leq -N. \end{aligned}$$

Let $R(t)$ be a bounded left continuous real monotone increasing function on $I = (-\infty, +\infty)$, and $F(t)$ a continuous real function on I . Suppose that the Stieltjes integrals

$$J_1 \equiv S \int_{-\infty}^{+\infty} F(t) dR(t), \quad J_2 \equiv S \int_{-\infty}^{+\infty} F(t) dR(t)$$

exist in the sense of finer divisions, i.e.,

$$(1) \quad J_1 = L_{\pi} \sum_{\Delta} F(t_{\Delta}) \Delta R(t), \quad J_2 = L_{\pi} \sum_{\Delta} |F(t_{\Delta})| \Delta R(t).$$

Here we can take $F(t_{\Delta})$ to be the maximum $\bar{F}_{\Delta}$ or minimum $\underline{F}_{\Delta}$ of $F(t)$ in Δ . Now for every $M > 0$ we have, due to continuity of $|F|_N$ on $[-M, +M]$,

$$LS \int_{-M}^{+M} |F|_N dR = S \int_{-M}^{+M} |F|_N dR \leq S \int_{-M}^{+M} |F| dR \leq S \int_{-\infty}^{+\infty} |F| dR = J_2,$$

the LS-integral being taken over $[-M, +M]$, so that

$$LS \int_{-\infty}^{+\infty} |F|_N dR = L_{M=+\infty} LS \int_{-M}^{+M} |F|_N dR \leq J_2.$$

It follows that $|F|$ and hence F are Lebesgue-Stieltjes integrable.

By the mean-value theorem for Lebesgue-Stieltjes integrals,

$$\underline{F}_{\Delta} \Delta R \leq \text{LS} \int_{\Delta_0} F dR \leq \overline{F}_{\Delta} \Delta R,$$

Δ_0 being the right semi-open interval of Δ without its right end-point. Consequently

$$\sum_{\Delta} \underline{F}_{\Delta} \Delta R \leq \sum_{\Delta} \text{LS} \int_{\Delta_0} F dR = \text{LS} \int_{-\infty}^{+\infty} F dR \leq \sum_{\Delta} \overline{F}_{\Delta} \Delta R,$$

which shows that

$$\text{LS} \int_{-\infty}^{+\infty} F dR = J_1,$$

since

$$\sum_{\Delta} \underline{F}_{\Delta} \Delta R \rightarrow J_1, \quad \sum_{\Delta} \overline{F}_{\Delta} \Delta R \rightarrow J_1,$$

by (1). Thus, we obtain

THEOREM 1. If $R(t)$ is a left continuous real monotone increasing function and $F(t)$ real-valued and continuous on $(-\infty, +\infty)$, and if the Stieltjes integrals

$$S \int_{-\infty}^{+\infty} F(t) dR(t), \quad S \int_{-\infty}^{+\infty} |F(t)| dR(t)$$

exist in the sense of finer divisions, then they exist also as Lebesgue-Stieltjes integrals and

$$S \int_{-\infty}^{+\infty} F(t) dR(t) = \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t).$$

Conversely, suppose that $F(t)$ is LS-integrable on $(-\infty, +\infty)$. Due to the complete additivity of an LS-integral,

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \sum_{\Delta_0} \text{LS} \int_{\Delta_0} F(t) dR(t).$$

Suppose $F(t) \geq 0$. Then the series

$$\sum_{\Delta} \underline{F}_{\Delta} \Delta R(t), \quad \underline{F}_{\Delta} \equiv \underline{B} F(t) \text{ in } \Delta,$$

converges to a finite sum, and

$$\sum_{\Delta} \bar{F}_{\Delta} \Delta R(t) \leq \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t).$$

On account of the uniform continuity of $F(t)$ there is a d_e such that

$$|\Delta F| \leq e \quad \text{for } \Delta \leq d_e.$$

Let π be a partition with $N\pi \leq d_e$. We have

$$\begin{aligned} 0 &\leq \sum_{\Delta} \frac{\pi}{\Delta} [F(t_{\Delta}) \Delta R(t) - \bar{F}_{\Delta} \Delta R(t)] = \sum_{\Delta} \frac{\pi}{\Delta} [F(t_{\Delta}) - \bar{F}_{\Delta}] \Delta R(t) \\ &\leq \sum_{\Delta} \frac{\pi}{\Delta} e \Delta R(t) \leq eN, \end{aligned}$$

from which follows

$$(2) \quad \sum_{\Delta} \frac{\pi}{\Delta} F(t_{\Delta}) \Delta R(t) \leq \sum_{\Delta} \frac{\pi}{\Delta} \bar{F}_{\Delta} \Delta R(t) + eN.$$

Similarly for $N\pi \leq d_e$

$$(3) \quad \sum_{\Delta} \frac{\pi}{\Delta} \bar{F}_{\Delta} \Delta R(t) - eN \leq \sum_{\Delta} \frac{\pi}{\Delta} F(t_{\Delta}) \Delta R(t).$$

But

$$\text{LS} \int_{\Delta_0} (\bar{F}_{\Delta} - F(t)) dR(t) \leq e \Delta R(t), \quad \text{LS} \int_{\Delta_0} (F(t) - \bar{F}_{\Delta}) dR(t) \leq e \Delta R(t).$$

Accordingly

$$\lim_{N\pi=0} \sum_{\Delta} \frac{\pi}{\Delta} \bar{F}_{\Delta} \Delta R(t) = \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \lim_{N\pi=0} \sum_{\Delta} \frac{\pi}{\Delta} F(t_{\Delta}) \Delta R(t),$$

and hence by (2), (3),

$$\lim_{N\pi=0} \sum_{\Delta} \frac{\pi}{\Delta} F(t_{\Delta}) \Delta R(t) = \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t).$$

By considering the positive and negative parts of a function $F(t)$ we readily prove

THEOREM 2. Under the first hypothesis of Theorem 1 and the condition that $F(t)$ is real, uniformly continuous and Lebesgue-Stieltjes integrable, we have

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \text{S} \int_{-\infty}^{+\infty} F(t) dR(t),$$

the Stieltjes integral on the right-hand side being taken in the sense of norm.

Consider an arbitrary system of bounded continuous real monotone increasing functions $\{R^{(\alpha)}(t)\}$ on $(-\infty, +\infty)$. Suppose $R(t)$ bounded continuous and the general sum

$$\sum^{\alpha} R^{(\alpha)}(t) = R(t)$$

for every t . Plainly then $R(t)$ is also a monotone increasing function and the series converges absolutely. Let E be a Borel measurable set with measures $\varphi_R(E)$, $\varphi_{R^{(\alpha)}}(E)$ relative to the functions $R(t)$, $R^{(\alpha)}(t)$. If D is a set of denumerably many non-overlapping intervals Δ_n , then

$$\varphi_{R^{(\alpha)}}(D) = \sum_{n=1}^{\infty} \Delta_n R^{(\alpha)}(t) = \overline{E} \sum_{n=1}^{\infty} \Delta_n R^{(\alpha)}(t), \quad D = \sum_{n=1}^{\infty} \Delta_n.$$

From the obvious equality

$$\Delta R(t) = \sum^{\alpha} \Delta R^{(\alpha)}(t) = \overline{E} \sum_{\sigma}^{\sigma} \Delta R^{(\alpha)}(t),$$

when σ is a finite set of the α 's we find

$$\begin{aligned} \varphi_R(D) &= \sum_{n=1}^{\infty} \Delta_n R(t) = \sum_{n=1}^{\infty} \sum^{\alpha} \Delta_n R^{(\alpha)}(t) = \overline{E} \overline{E} \sum_{n=1}^{\infty} \sum_{\sigma}^{\sigma} \Delta_n R^{(\alpha)}(t) \\ &= \overline{E} \overline{E} \sum_{\sigma}^{\sigma} \sum_{n=1}^{\infty} \Delta_n R^{(\alpha)}(t) = \overline{E} \sum_{\sigma}^{\sigma} \varphi_{R^{(\alpha)}}(D) = \sum^{\alpha} \varphi_{R^{(\alpha)}}(D). \end{aligned}$$

Now, the complementary set $\overline{E}$ is also Borel measurable, and there exists a sequence of sets D_n such that

$$\overline{E} \subset D_n, \quad D_{n+1} \subset D_n, \quad \varphi_R(D_n) \rightarrow \varphi_R(\overline{E}).$$

From $\varphi_R(D_n - \overline{E}) = \varphi_R(D_n) - \varphi_R(\overline{E})$, we secure

$$\varphi_{R(\alpha)}(D_n) \rightarrow \varphi_{R(\alpha)}(\bar{E}),$$

since

$$\varphi_{R(\alpha)}(D_n) - \varphi_{R(\alpha)}(\bar{E}) = \varphi_{R(\alpha)}(D_n - \bar{E}),$$

$$\varphi_{R(\alpha)}(D_n - \bar{E}) \leq \varphi_{R(D_n - \bar{E})}.$$

It follows that

$$\begin{aligned} \bar{E}_n [\varphi_{R(I)} - \varphi_{R(D_n)}] &= \bar{E}_n \bar{E}_\sigma \sum_{\alpha}^{\sigma} [\varphi_{R(\alpha)}(I) - \varphi_{R(\alpha)}(D_n)] \\ &= \bar{E}_\sigma \sum_{\alpha}^{\sigma} \bar{E}_n [\varphi_{R(\alpha)}(I) - \varphi_{R(\alpha)}(D_n)] \\ &= \bar{E}_\sigma \sum_{\alpha}^{\sigma} [\varphi_{R(\alpha)}(I) - \varphi_{R(\alpha)}(\bar{E})] \\ &= \bar{E}_\sigma \sum_{\alpha}^{\sigma} \varphi_{R(\alpha)}(I - \bar{E}) = \sum_{\alpha}^{\sigma} \varphi_{R(\alpha)}(\bar{E}). \end{aligned}$$

Thus

$$(4) \quad \varphi_R(\bar{E}) = \sum_{\alpha}^{\sigma} \varphi_{R(\alpha)}(\bar{E}).$$

To extend this result to integrals we observe that

$$\begin{aligned} \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) &= \text{L} \sum_{n=\infty}^{\pi_n} \chi_m \varphi_R(E_m), \\ (5) \quad \text{LS} \int_{-\infty}^{+\infty} F(t) dR^{(\alpha)}(t) &= \text{L} \sum_{n=\infty}^{\pi_n} \chi_m \varphi_{R(\alpha)}(E_m), \end{aligned}$$

where $F(t)$ is a bounded positive Borel measurable function, and

$$N\pi_n \leq 1/n, \quad E_m \equiv [\chi_m \leq F(t) < \chi_{m+1}].$$

Further, as we have used χ_m which is the smallest value that $F(t)$ can take in E_m , the sequence

$$\sum_{n=\infty}^{\pi_n} \chi_m \varphi_R(E_m)$$

is monotone increasing in n . Consequently, by (4), (5),

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \bar{E} \sum_{n=\infty}^{\pi_n} \chi_m \varphi_R(E_m) = \bar{E} \sum_{n=\infty}^{\pi_n} \chi_m \left[\bar{E}_\sigma \sum_{\alpha}^{\sigma} \varphi_{R(\alpha)}(E_m) \right]$$

$$\begin{aligned}
&= \frac{1}{n} \frac{1}{\sigma} \sum_{\alpha}^{\pi_n} \sum_{\alpha}^{\sigma} \chi_m \varphi_{R(\alpha)}(E_m) = \frac{1}{\sigma} \frac{1}{n} \sum_{\alpha}^{\sigma} \sum_{\alpha}^{\pi_n} \chi_m \varphi_{R(\alpha)}(E_m) \\
&= \frac{1}{\sigma} \sum_{\alpha}^{\sigma} \text{LS} \int_{-\infty}^{+\infty} F(t) dR^{(\alpha)}(t).
\end{aligned}$$

Hence

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \sum_{\alpha} \text{LS} \int_{-\infty}^{+\infty} F(t) dR^{(\alpha)}(t).$$

For an unbounded positive integrable function this still holds, again by considering the upper bounds. Evidently one can extend this equality to any integrable function, using the positive and negative parts. We are then led to

THEOREM 3. Let $R^{(\alpha)}(t)$ be an arbitrary system of real, bounded, continuous monotone increasing functions, whose general sum $R(t)$ is also bounded and continuous. Then for any real function $F(t)$ integrable relative to $R(t)$ the following equality holds:

$$(6) \quad \text{LS} \int_{-\infty}^{+\infty} F(t) dR(t) = \sum_{\alpha} \text{LS} \int_{-\infty}^{+\infty} F(t) dR^{(\alpha)}(t).$$

2. Strong convergence of certain functional integrals

THEOREM 4. Given a vector ξ of $\mathcal{M}$ and a real Borel measurable function $F(t)$. A necessary and sufficient condition¹ for the existence, as a strong limit, of the functional Lebesgue-Stieltjes integral of $F(t)$ with respect to $E_t \xi$ is that the numerical Lebesgue-Stieltjes integral of $|F(t)|^2$ with respect to $(E_t \xi, \xi)$ exists as a finite limit. If this condition is fulfilled,

$$(\text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \xi, \mu) = \text{LS} \int_{-\infty}^{+\infty} F(t) d(E_t \xi, \mu),$$

$$\| \text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \xi \|^2 = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi).$$

¹Cf. Stone's third Note.

The condition is necessary. For, when the functional integral exists as a strong limit,

$$\begin{aligned} \left\| \text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \xi \right\|^2 &= \lim_{N \rightarrow +\infty} \left\| \text{LS} \int_{-\infty}^{+\infty} F_N(t) dE_t \xi \right\|^2 \\ &= \lim_{N \rightarrow +\infty} \text{LS} \int_{-\infty}^{+\infty} |F_N(t)|^2 d(E_t \xi, \xi) \\ &= \lim_{N \rightarrow +\infty} \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi) < +\infty, \end{aligned}$$

where $F_N(t)$ is the truncated function of $F(t)$, as introduced in the proof of Theorem 1. Now, the last limit is, by definition, the numerical integral of $|F(t)|^2$ relative to $(E_t \xi, \xi)$, so that the integral exists as a finite limit. This also establishes the second equality in the theorem. The first equality is a direct consequence of the strong convergence of the functional integral.

The condition is also sufficient. Since the numerical integral exists as a finite limit,

$$(7) \quad \lim_{N \rightarrow +\infty} \text{LS} \int_{-\infty}^{+\infty} |F_N(t)|^2 d(E_t \xi, \xi) = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi).$$

On the other hand,

$$|F_N(t) \cdot F(t)| \leq |F(t)|^2, \quad \lim_{N \rightarrow +\infty} F_N(t) = F(t),$$

and hence

$$\lim_{N \rightarrow +\infty} \text{LS} \int_{-\infty}^{+\infty} F_N(t) F(t) d(E_t \xi, \xi) = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi),$$

which, on account of (7), gives

$$(8) \quad \lim_{N \rightarrow +\infty} \text{LS} \int_{-\infty}^{+\infty} (F_N(t) - F(t)) d(E_t \xi, \xi) = 0.$$

The fact that the numerical integral of $|F(t)|^2$ takes a finite value shows that the positive and negative parts F^+ , F^- of F are also quadratically integrable, because

$$F^+ \geq 0, \quad F^- \leq 0, \quad |F|^2 = (F^+ - F^-)^2 = |F^+|^2 + |F^-|^2 - 2F^+F^-, \\ F^+F^- = 0.$$

So, we need consider only positive functions. For a positive $F(t)$ we have

$$(F_M - F_N)^2 = ((F_M - F) + (F - F_N))^2 \leq (F_M - F)^2 + (F - F_N)^2,$$

as $(F_M - F)(F - F_N) \leq 0$. Consequently,

$$\text{LS} \int_{-\infty}^{+\infty} (F_M - F_N)^2 d(E_t \xi, \xi) \leq \text{LS} \int_{-\infty}^{+\infty} (F_M - F)^2 d(E_t \xi, \xi) + \\ \text{LS} \int_{-\infty}^{+\infty} (F - F_N)^2 d(E_t \xi, \xi) \\ \leq e + e,$$

for sufficiently large M and N , by (8). Then from the equality

$$\left\| \text{LS} \int_{-\infty}^{+\infty} (F_M - F_N) dE_t \xi \right\|^2 = \text{LS} \int_{-\infty}^{+\infty} (F_M - F_N) d(E_t \xi, \xi),$$

we infer that the functional integral in question exists as a strong limit, since the Cauchy condition is now satisfied.

It may be noted that the functional integral connects three things, $F(t)$, Ω , ξ . It defines a vector, say, $F(\Omega)\xi$. One naturally asks: When will $F(\Omega)$ be self-adjoint? A simple answer may be given to this. An operatorial function $F(\Omega)$ is self-adjoint, if and only if the operatorial measure (relative to E_t) of the set where $F(t)$ is finite coincides with the identity operator. (Operatorial measure of sets has been introduced in Chapter II.) A detailed discussion of this matter will appear in the near future.

THEOREM 5. If the condition

$$(9) \quad \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi) < +\infty$$

holds for a vector ξ in $\mathcal{M}$ and a continuous function $F(t)$, the integral being taken in the sense of finer divisions, then the functional integrals

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \xi, \quad \text{S} \int_{-\infty}^{+\infty} F(t) dE_t \xi$$

exist as strong limits in the sense of norm and they are equal.

Applying Theorem 1 to $MF(t)|^2$ we see that

$$J \equiv \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi) = \text{S} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \xi, \xi) < +\infty,$$

and hence by Theorem 4, the functional integral

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \xi$$

exists as a strong limit and defines a vector ξ^* . Clearly there exists an $M_e > 0$ such that

$$(10) \quad \left\| \xi^* - \text{S} \int_{-M_e+1}^{+M_e-1} F(t) dE_t \xi \right\| \leq e.$$

Keep this M_e fixed. Due to the orthogonal properties of E_t ,

$$(11) \quad \begin{aligned} \text{S} \int_{-\infty}^{-M_e+1} |F(t)|^2 d(E_t \xi, \xi) &= \left\| \text{LS} \int_{(-\infty, -M_e+1)} F(t) dE_t \xi \right\|^2 \leq e^2, \\ \text{S} \int_{+M_e-1}^{+\infty} |F(t)|^2 d(E_t \xi, \xi) &= \left\| \text{LS} \int_{(+M_e-1, +\infty)} F(t) dE_t \xi \right\|^2 \leq e^2. \end{aligned}$$

Because of (9) there are partitions π of $(-\infty, +\infty)$ with $N\pi \leq d_e(1) < 1$ and

$$\left| J - \sum_{\Delta} \frac{\pi}{\Delta} |F(t_{\Delta})|^2 \Delta(E_t \xi, \xi) \right| \leq e.$$

This implies that

$$\sum_{\Delta} \frac{\pi}{\Delta} F(t_{\Delta}) \Delta E_t \xi$$

converges strongly. In π there are only two intervals containing the points $-M_e, M_e$. Call these intervals $\Delta_{\pi}, \delta_{\pi}$ respectively.

Let $\Delta\pi \equiv [a'_\pi, a_\pi]$, $\delta_\pi \equiv [b_\pi, b'_\pi]$, $\Delta''_\pi \equiv [a'_\pi, -M_e]$, $\Delta'_\pi \equiv [-M_e, a_\pi]$, $\delta'_\pi \equiv [b_\pi, +M_e]$, $\delta''_\pi \equiv [+M_e, b'_\pi]$. Then

$$\begin{aligned}
 \sum_{\Delta} \frac{\pi}{\Delta} F(t_\Delta) \Delta E_t \xi &= \sum_{\Delta\pi}^{\langle -\infty, a'_\pi \rangle} F(t_\Delta) \Delta E_t \xi + F(t_{\Delta\pi}) \left[\Delta''_\pi E_t \xi + \Delta'_\pi E_t \xi \right] \\
 &\quad + \sum_{\Delta\pi}^{\langle a_\pi, b_\pi \rangle} F(t_\Delta) \Delta E_t \xi + F(t_{\delta_\pi}) \delta_\pi E_t \xi \\
 &\quad + \sum_{\Delta\pi}^{\langle b'_\pi, +\infty \rangle} F(t_\Delta) \Delta E_t \xi \\
 (12) \quad &= \left(\sum_{\Delta\pi}^{\langle -\infty, a'_\pi \rangle} + F(t_{\Delta\pi}) \Delta''_\pi E_t \xi \right) \\
 &\quad + \left[F(t_{\Delta\pi}) \Delta'_\pi E_t \xi + \sum_{\Delta\pi}^{\langle a_\pi, b_\pi \rangle} + F(t_{\delta_\pi}) \delta'_\pi E_t \xi \right] \\
 &\quad + \left\{ F(t_{\delta_\pi}) \delta''_\pi E_t \xi + \sum_{\Delta\pi}^{\langle b'_\pi, +\infty \rangle} \right\}
 \end{aligned}$$

However, by the orthogonal relations of ΔE_t ,

$$\begin{aligned}
 &\left\| \sum_{\Delta\pi}^{\langle -\infty, a'_\pi \rangle} F(t_\Delta) \Delta E_t \xi + F(t_{\Delta\pi}) \Delta''_\pi E_t \xi \right\|^2 \\
 (13) \quad &\leq \sum_{\Delta\pi}^{\langle -\infty, a'_\pi \rangle} |F(t_\Delta)|^2 \|\Delta E_t \xi\|^2 + |F(t_{\Delta\pi})|^2 \|\Delta''_\pi E_t \xi\|^2 \\
 &\leq \sum_{\Delta\pi}^{\langle -\infty, a'_\pi \rangle} |F(t_\Delta)|^2 \|\Delta E_t \xi\|^2 + |F(t_{\Delta\pi})|^2 \|\Delta\pi E_t \xi\|^2 \\
 &\leq S \int_{-\infty}^{a_\pi} |F(t)|^2 d(E_t \xi, \xi) + e \leq e^2 + e,
 \end{aligned}$$

from (11), when $N\pi \leq d_e(2) < d_e(1) < 1$. Similarly,

$$(14) \quad \left\| F(t_{\delta_\pi}) \delta''_\pi E_t \xi + \sum_{\Delta\pi}^{\langle b'_\pi, +\infty \rangle} F(t_\Delta) \Delta E_t \xi \right\|^2 \leq e^2 + e.$$

From (12) and the triangle inequality we secure

$$(15) \quad \left\| S \int_{-M_e}^{+M_e} F(t) dE_t \xi - \sum_{\Delta} \frac{\pi}{\Delta} F(t_\Delta) \Delta E_t \xi \right\| \leq \left\| S \int_{-M_e}^{+M_e} -[\] \right\| + \left\| (\) \right\| + \left\| \{ \} \right\|.$$

But

$$\begin{aligned}
 \left\| S \int_{-M_e}^{+M_e} -[\] \right\| &\leq \left\| S \int_{-M_e}^{+M_e} - \left(F(t_{\Delta'_\pi}) \Delta'_\pi E_t \xi + \sum_{\Delta\pi}^{\langle a_\pi, b_\pi \rangle} + F(t_{\delta'_\pi}) \delta'_\pi E_t \xi \right) \right\| \\
 &\quad + \left\| (F(t_{\Delta_\pi}) - F(t_{\Delta'_\pi})) \Delta'_\pi E_t \xi + (F(t_{\delta_\pi}) - F(t_{\delta'_\pi})) \delta'_\pi E_t \xi \right\|.
 \end{aligned}$$

The first term on the right hand side can be made less than e , if $N\pi \leq d_e^{(3)}$, in approximating the integral over $[-M_e, +M_e]$.

The second term also becomes less than e when $N\pi \leq d_e^{(4)}$, as $F(t)$ is uniformly continuous in $[-M_e - 1, +M_e + 1]$. For $N\pi \leq d_e$, which is the smaller of $d_e^{(2)}$, $d_e^{(3)}$, $d_e^{(4)}$, we have then

$$\left\| S \int_{-M_e}^{+M_e} F(t) dE_t \xi - \sum_{\Delta} F(t_{\Delta}) \Delta E_t \xi \right\| \leq 2e,$$

so that, by (15), (13), (14),

$$(16) \quad \left\| S \int_{-M_e}^{+M_e} F(t) dE_t \xi - \sum_{\Delta} F(t_{\Delta}) \Delta E_t \xi \right\| \leq 2e + 2e + 2e.$$

Now, from the simple inequality

$$\left\| \xi^* - \sum_{\Delta} \right\| \leq \left\| \xi^* - S \int_{-M_e}^{+M_e} \right\| + \left\| S \int_{-M_e}^{+M_e} - \sum_{\Delta} \right\|,$$

we derive immediately

$$\left\| \xi^* - \sum_{\Delta} F(t_{\Delta}) \Delta E_t \xi \right\| \leq 2e + 6e,$$

by virtue of (11) and (16). This means precisely that the functional Stieltjes integral of $F(t)$ with respect to $E_t \xi$ exists as a strong limit and it is equal to ξ^* , the corresponding Lebesgue-Stieltjes integral.

In the remainder of this section we consider a one-parameter family of vectors $\{\xi_t\}$ with the following properties:

(17) ξ_t is strongly continuous in t .

(18) $(\xi_s, \xi_t) = (\xi_s, \xi_s)$ for $s \leq t$.

(19) $\|\xi_t\|$ is bounded on $(-\infty, +\infty)$.

THEOREM 6. For a real continuous function $G(t)$ on $[a, b]$ the integral

$$\int_a^b G(t) \frac{d(\xi, \xi_t) d\xi_t}{d\|\xi_t\|^2} \equiv \lim_{N\pi=0} \sum_{\Delta} G(t_{\Delta}) \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}$$

exists and is equal to

$$S \int_a^b G(t) d \int_a^t \frac{d(\xi, \xi_t) d \xi_t}{d \|\xi_t\|^2}.$$

By the additivity of Hellinger integrals,

$$\sum_{\Delta} \pi \Delta H_{\xi} = H_{\xi} \equiv \int_a^b \frac{d(\xi, \xi_t) d \xi_t}{d \|\xi_t\|^2}, \quad \Delta H_{\xi} \equiv \int_{\Delta} \frac{d(\xi, \xi_t) d \xi_t}{d \|\xi_t\|^2},$$

where π is a partition of $[a, b]$. Evidently

$$\lim_{N\pi=0} \sum_{\Delta} \pi \Delta H_{\xi} = H_{\xi},$$

and hence

$$\begin{aligned} (20) \quad \|H_{\xi}\|^2 &= \lim_{N\pi=0} \left\| \sum_{\Delta} \pi \Delta H_{\xi} \right\|^2 = \lim_{N\pi=0} \left(\sum_{\Delta} \pi \Delta H_{\xi}, \sum_{\Delta} \pi \Delta H_{\xi} \right) \\ &= \lim_{N\pi=0} \sum_{\Delta} \pi (\Delta H_{\xi}, \Delta H_{\xi}), \end{aligned}$$

because of (18). On the other hand, by the theorem on functional Hellinger integrals in Chapter II,

$$\lim_{N\pi=0} \sum_{\Delta} \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} = H_{\xi},$$

so that

$$\begin{aligned} (21) \quad \|H_{\xi}\|^2 &= \lim_{N\pi=0} \left(\sum_{\Delta} \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}, \sum_{\Delta} \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right) \\ &= \lim_{N\pi=0} \sum_{\Delta} \left(\frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}, \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right). \end{aligned}$$

Now for $\Delta \equiv (\underline{\Delta}, \bar{\Delta})$,

$$\left(\left(\int_a^{\underline{\Delta}} + \int_{\underline{\Delta}}^{\bar{\Delta}} + \int_{\bar{\Delta}}^b \right) \frac{d(\xi, \xi_t) d \xi_t}{d \|\xi_t\|^2}, \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right) = \left(\int_{\underline{\Delta}}^{\bar{\Delta}} \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right).$$

Thus

$$(H_{\xi}, \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}) = (\Delta H_{\xi}, \frac{\Delta(\xi, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}),$$

which gives

$$\begin{aligned}
 \|H_{\frac{1}{2}}\|^2 &= (H_{\frac{1}{2}}, \frac{1}{N} \sum_{n=0}^N \frac{\pi}{\Delta} \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}) \\
 (22) \quad &= \frac{1}{N} \sum_{n=0}^N \frac{\pi}{\Delta} (\Delta H_{\frac{1}{2}}, \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}).
 \end{aligned}$$

From (20), (21), (22) we infer at once that

$$\begin{aligned}
 (23) \quad \frac{1}{N} \sum_{n=0}^N \left\| \Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right\|^2 &= \|H_{\frac{1}{2}}\|^2 + \|H_{\frac{1}{2}}\|^2 - \\
 &(\|H_{\frac{1}{2}}\|^2 + \|H_{\frac{1}{2}}\|^2) = 0.
 \end{aligned}$$

As

$$\begin{aligned}
 &\left\| \frac{\pi}{\Delta} G(t_{\Delta}) (\Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}) \right\|^2 \\
 &= \frac{\pi}{\Delta} |G(t_{\Delta})|^2 \left\| \Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right\|^2, \\
 |G(t_{\Delta})|^2 \left\| \Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right\|^2 &\leq E |G(t)|^2 \cdot \left\| \Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2} \right\|^2,
 \end{aligned}$$

we have by (23),

$$(24) \quad \frac{1}{N} \sum_{n=0}^N \left[G(t_{\Delta}) (\Delta H_{\frac{1}{2}} - \frac{\Delta(\frac{1}{2}, \xi_t) \Delta \xi_t}{\Delta \|\xi_t\|^2}) \right] = 0.$$

But the Stieltjes integral

$$S \int_a^b G(t) d \int_a^t \frac{d(\frac{1}{2}, \xi_t) d \xi_t}{d \|\xi_t\|^2} \equiv \frac{1}{N} \sum_{n=0}^N G(t_{\Delta}) \Delta H_{\frac{1}{2}}$$

exists,¹ so that (24) implies

$$\int_a^b G(t) \frac{d(\frac{1}{2}, \xi_t) d \xi_t}{d \|\xi_t\|^2} = S \int_a^b G(t) d \int_a^t \frac{d(\frac{1}{2}, \xi_t) d \xi_t}{d \|\xi_t\|^2}.$$

THEOREM 7. If $F(t)$ is a real continuous function for which the Stieltjes integral

$$S \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\frac{1}{2}, \xi_t) d \xi_t}{d \|\xi_t\|^2}$$

¹This can be easily proved by utilizing the uniform continuity of $G(t)$ in $[a, b]$, as in the usual numerical case.

exists in the strong sense of norm, then the Hellinger integral

$$\int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t) d\xi_t}{d\|\xi_t\|^2}$$

exists as a strong limit in the sense of norm, and the two integrals are equal.

Divide the infinite interval $(-\infty, +\infty)$ into infinitely many non-overlapping finite intervals $[\Delta_n]$. Then

$$\sum_{n=1}^{\infty} S \int_{\Delta_n} F(t) dH_t \xi = S \int_{-\infty}^{+\infty} F(t) dH_t \xi, \quad H_t \xi \equiv \int_{-\infty}^t \frac{d(\xi, \xi_t) d\xi_t}{d\|\xi_t\|^2}.$$

It follows that there exist α_e, β_e such that

$$\left\| S \int_{-\infty}^{+\infty} F(t) dH_t \xi - S \int_{\alpha_e+1}^{\beta_e-1} F(t) dH_t \xi \right\|^2 \leq e^2.$$

Consequently

$$(25) \quad \left\| S \int_{-\infty}^{\alpha_e+1} F(t) dH_t \xi \right\|^2 \leq e^2, \quad \left\| S \int_{\beta_e-1}^{+\infty} F(t) dH_t \xi \right\|^2 \leq e^2.$$

For the interval $I_e \equiv [\alpha_e, \beta_e]$ there is a $d_e^{(1)}$ such that

$$(26) \quad \left\| S \int_{\Delta_e} F(t) dH_t \xi - \sum_{\Delta \subset I_e} F(t_{\Delta}) J_{\Delta} \xi \right\|^2 \leq e^2, \quad J_{\Delta} \xi \equiv \frac{d(\xi, \xi_{\Delta}) d\xi_{\Delta}}{d\|\xi_{\Delta}\|^2},$$

for any π with $N\pi \leq d_e^{(1)} < 1$, by Theorem 6.

As ξ_t is continuous, $H_t \xi$ is also continuous and we can find a $d_e^{(2)}$ so small that when $\Delta_{\alpha_e} \leq d_e^{(2)}$, $\Delta_{\beta_e} \leq d_e^{(2)}$,

$$\left\| \Delta_{\alpha_e} H_t \xi \right\| \leq e/(\bar{E} + 1), \quad \left\| \Delta_{\beta_e} H_t \xi \right\| \leq e/(\bar{E} + 1),$$

where $\Delta_{\alpha_e}, \Delta_{\beta_e}$ are intervals containing α_e, β_e respectively, and $\bar{E}_e$ is the maximum of $|F(t)|$ in $[\alpha_e - 1, \beta_e + 1]$. So,

$$(27) \quad \left\| F(t_{\Delta_{\alpha_e}}) J_{\Delta_{\alpha_e}} \xi \right\| \leq e, \quad \left\| F(t_{\Delta_{\beta_e}}) J_{\Delta_{\beta_e}} \xi \right\| \leq e$$

for $\Delta_{\alpha_e} \leq d_e^{(2)}$, $\Delta_{\beta_e} \leq d_e^{(2)}$.

Now consider a partition π of $(-\infty, +\infty)$. In π there is one and only one pair of intervals containing α_e, β_e . Let the end-points of Δ_{α_e} be α_π, α'_π so that $\Delta_{\alpha_e} = [\alpha_\pi, \alpha'_\pi]$. Similarly $\Delta_{\beta_e} = [\beta'_\pi, \beta_\pi]$. Then for π with $N\pi \leq d_e^{(3)} \equiv$ the smaller of $d_e^{(1)}, d_e^{(2)}$,

$$(28) \quad \left\| \int_{\alpha_\pi}^{\alpha'_\pi} F(t) dH_t \right\|^2 \leq e^2, \quad \left\| \int_{\beta'_\pi}^{\beta_\pi} F(t) dH_t \right\|^2 \leq e^2,$$

by (25), since Δ_{α_e} lies in $(-\infty, \alpha_e + 1]$ and Δ_{β_e} in $[\beta_e - 1, +\infty)$. On the other hand, the intervals of π in $I' \equiv [\alpha'_\pi, \beta'_\pi]$ and the intervals $I_\alpha \equiv [\alpha_e, \alpha'_\pi]$, $I_\beta \equiv [\beta'_\pi, \beta_e]$ together form a partition π_e^* of I_e , for which we have

$$\begin{aligned} \left\| \int_{\alpha_e}^{\beta_e} F(t) dH_t - \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 &= \left\| \left(\int_{\alpha_e}^{\alpha'_\pi} + \int_{\alpha'_\pi}^{\beta'_\pi} + \int_{\beta'_\pi}^{\beta_e} \right) F(t) dH_t \right. \\ &\quad \left. - (F(t_{I_\alpha}) J_{I_\alpha} + \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta + F(t_{I_\beta}) J_{I_\beta}) \right\|^2 \\ &= \left\| \int_{\alpha_e}^{\alpha'_\pi} F(t) dH_t - F(t_{I_\alpha}) J_{I_\alpha} \right\|^2 + \left\| \int_{\alpha'_\pi}^{\beta'_\pi} F(t) dH_t \right. \\ &\quad \left. - \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 + \left\| \int_{\beta'_\pi}^{\beta_e} F(t) dH_t - F(t_{I_\beta}) J_{I_\beta} \right\|^2. \end{aligned}$$

Thus

$$(29) \quad \left\| \int_{\alpha'_\pi}^{\beta'_\pi} F(t) dH_t - \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 \leq \left\| \int_{\alpha_e}^{\beta_e} F(t) dH_t - \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 \leq e^2$$

for π with $N\pi \leq d_e^{(3)}$, by (26). But

$$\begin{aligned} \left\| \int_{\alpha_\pi}^{\beta_\pi} F(t) dH_t - \sum_{\Delta \in [\alpha_\pi, \beta_\pi]} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 &= \left\| \left(\int_{\alpha_\pi}^{\alpha'_\pi} + \int_{\alpha'_\pi}^{\beta'_\pi} + \int_{\beta'_\pi}^{\beta_\pi} \right) F(t) dH_t \right. \\ &\quad \left. - (F(t_{\Delta_{\alpha_e}}) J_{\Delta_{\alpha_e}} + \sum_{\Delta \in I'} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta + F(t_{\Delta_{\beta_e}}) J_{\Delta_{\beta_e}}) \right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \left\| S \int_{\alpha'_\pi}^{\beta'_\pi} F(t) dH_t \xi - \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \xi \right\|^2 + \left\| S \int_{\alpha_\pi}^{\alpha'_\pi} F(t) dH_t \xi \right. \\
&\quad \left. - F(t_{\Delta_{\alpha_e}}) J_{\Delta_{\alpha_e}} \xi \right\|^2 + \left\| S \int_{\beta_\pi}^{\beta'_\pi} F(t) dH_t \xi - F(t_{\Delta_{\beta_e}}) J_{\Delta_{\beta_e}} \xi \right\|^2 \\
&\leq e^2 + (e + e)^2 + (e + e)^2,
\end{aligned}$$

by virtue of (29), (28), (27). Consequently,

$$(30) \quad \left\| S \int_{\alpha_\pi}^{\beta_\pi} F(t) dH_t \xi - \sum_{\Delta \in [L_\pi, L_\pi]} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \xi \right\| \leq 3e$$

for π with $N\pi \leq d_e(3)$.

We note that there exists a $d_e(4)$ such that

$$\begin{aligned}
\left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) \Delta H_t \xi \right\|^2 &\leq \left\| S \int_{-\infty}^{\alpha_e+1} F(t) dH_t \xi \right\|^2 + e, \quad I_1 \equiv (-\infty, \alpha_e+1], \\
\left\| \sum_{\Delta \in I_2} \frac{\pi}{\Delta} F(t_\Delta) \Delta H_t \xi \right\|^2 &\leq \left\| S \int_{\beta_e-1}^{+\infty} F(t) dH_t \xi \right\|^2 + e, \quad I_2 \equiv [\beta_e-1, +\infty),
\end{aligned}$$

for π with $N\pi \leq d_e(4)$. Let d_e be the smaller of $d_e(3)$, $d_e(4)$.

Then (25) gives

$$(31) \quad \left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) \Delta H_t \xi \right\|^2 \leq e^2 + e, \quad \left\| \sum_{\Delta \in I_2} \frac{\pi}{\Delta} F(t_\Delta) \Delta H_t \xi \right\|^2 \leq e^2 + e.$$

By means of the Cauchy-Lagrange inequality we find

$$\left\| \Delta H_t \xi \right\|^2 = \int_{\Delta} \frac{d(\xi, \xi_t) d(\xi_t, \xi)}{d \|\xi_t\|^2} \geq \frac{\Delta(\xi, \xi_t) \Delta(\xi_t, \xi)}{\Delta \|\xi_t\|^2},$$

so that

$$(32) \quad \left\| \Delta H_t \xi \right\|^2 \geq \left\| J_\Delta \xi \right\|^2 = (1/(\Delta \xi_t, \Delta \xi_t)) \Delta(\xi, \xi_t) \Delta(\xi_t, \xi) \cdot \|\Delta \xi_t\|^2,$$

if we recall that because of (18), $\Delta \|\xi_t\|^2 = (\Delta \xi_t, \Delta \xi_t)$. This inequality implies that

$$\begin{aligned}
\left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \xi \right\|^2 &= \sum_{\Delta \in I_1} \frac{\pi}{\Delta} \left\| F(t_\Delta) J_\Delta \xi \right\|^2 \leq \sum_{\Delta \in I_1} \frac{\pi}{\Delta} \left\| F(t_\Delta) \Delta H_t \xi \right\|^2 \\
&= \left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) \Delta H_t \xi \right\|^2
\end{aligned}$$

which shows that for $N\pi \leq d_e$

$$(33) \quad \left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 \leq e^2 + e,$$

from (31). Similarly,

$$(34) \quad \left\| \sum_{\Delta \in I_2} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\|^2 \leq e^2 + e.$$

Now we can bring the proof to conclusion. From the obvious inequalities

$$\begin{aligned} & \left\| S \int_{-\infty}^{+\infty} F(t) dH_t \right\| - \left\| \sum_{\Delta} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\| \\ & \leq \left\| S \int_{-\infty}^{+\infty} - \sum_{\Delta \in [a_\pi, b_\pi]} \frac{\pi}{\Delta} \right\| + \left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} \right\| + \left\| \sum_{\Delta \in I_2} \frac{\pi}{\Delta} \right\| \\ & \leq \left(\left\| S \int_{-\infty}^{+\infty} - S \int_{a_\pi}^{\beta_\pi} \right\| + \left\| S \int_{a_\pi}^{\beta_\pi} - \sum_{\Delta \in [a_\pi, b_\pi]} \frac{\pi}{\Delta} \right\| \right) \\ & \quad + \left\| \sum_{\Delta \in I_1} \frac{\pi}{\Delta} \right\| + \left\| \sum_{\Delta \in I_2} \frac{\pi}{\Delta} \right\| \end{aligned}$$

we deduce immediately by (25), (30), and (33), (34),

$$\left\| S \int_{-\infty}^{+\infty} F(t) dH_t \right\| - \left\| \sum_{\Delta} \frac{\pi}{\Delta} F(t_\Delta) J_\Delta \right\| \leq (2e + 3e) + 3e + 3e,$$

provided $N\pi \leq d_e$.

3. Characteristic expansions

Let Ω be self-adjoint. According to a theorem in Chapter I the space $\mathcal{H}$ has an orthonormal basis $[\zeta^{(\delta)} / \delta]$. Now consider only those ζ 's with $C_r \zeta$ not identically zero in r . Well order these and obtain the system $[\zeta^{(\alpha)} / \alpha]$. Define by transfinite induction

$$(35) \quad \zeta_t^{(0)} \equiv \zeta_t^{(0)}, \quad \zeta_t^{(\alpha)} \equiv \zeta_t^{(\alpha)} - Q_t^{(\alpha)} \zeta_t^{(\alpha)}$$

where

$$(36) \quad \zeta_t^{(\alpha)} \equiv C_t \zeta^{(\alpha)}, \quad Q_t^{(\alpha)} \equiv P_{\mathcal{M}_t^{(\alpha)}} \zeta_t^{(\alpha)}, \quad \mathcal{M}_t^{(\alpha)} \equiv \sum_{\beta < \alpha} \mathcal{M}_t^{(\beta)},$$

and $\mathcal{M}_t^{(\alpha)}$ is the set of all vectors $\zeta_s^{(\alpha)}$ with $s \leq t$.

We proceed to show that the every system $\{\zeta_t^{(\alpha)} | t\}$ has the following property:

$$(37) \quad C_t \zeta_r^{(\alpha)} = \zeta_s^{(\alpha)}, \quad s \equiv \text{the smaller of } t, r.$$

Obviously $\{\zeta_t^{(0)}\}$ satisfies this condition. To make an induction argument let us assume that (37) holds for every $\{\zeta_t^{(\beta)} | t\}$,

$\beta < \alpha$. We first observe from definition (35) that

$$(38) \quad (\zeta_t^{(\beta)}, \zeta_t^{(\gamma)}) = 0, \quad \beta \neq \gamma, \quad \beta < \alpha, \quad \gamma < \alpha,$$

by virtue of the orthogonal decomposition theorem in Chapter I.

Moreover, for $s \equiv$ smaller of t, r ,

$$\begin{aligned} (\zeta_r^{(\beta)}, \zeta_t^{(\gamma)}) &= (C_r \zeta_r^{(\beta)}, C_t \zeta_t^{(\gamma)}) = (C_t C_r \zeta_r^{(\beta)}, \zeta_t^{(\gamma)}) \\ &= (C_s \zeta_r^{(\beta)}, \zeta_t^{(\gamma)}) = (\zeta_s^{(\beta)}, C_s \zeta_t^{(\gamma)}) \\ &= (\zeta_s^{(\beta)}, \zeta_s^{(\gamma)}), \end{aligned}$$

by property (37). But from (38) we infer that $\zeta_r^{(\beta)}, \zeta_t^{(\gamma)}$ are always orthogonal for any pair of values r, t , whenever $\beta \neq \gamma$, $\beta < \alpha$, $\gamma < \alpha$. We have then the further property:

$$(39) \quad \{\zeta_r^{(\beta)} | r\}, \quad \{\zeta_t^{(\gamma)} | t\} \text{ are orthogonal systems.}$$

On the other hand, (38) gives

$$\zeta_x^{(\beta)} = C_x \zeta_t^{(\beta)}, \quad x \leq t.$$

For a given x_0 take $t_0 \equiv x_0 + 1$. Then

$$\zeta_x^{(\beta)} = C_x \zeta_{t_0}^{(\beta)}, \quad x \leq x_0 + 1.$$

But C_x is strongly continuous. Hence $C_x \zeta_{t_0}^{(\beta)}$ is continuous at

x_0 . So, $\xi_x^{(\beta)}$ is continuous at x_0 , and

$$(40) \quad \xi_t^{(\beta)} \text{ is strongly continuous in } t, \quad \beta < \alpha.$$

Moreover, as

$$\xi_x^{(\beta)} = c_x \xi_1^{(\beta)}$$

for $x \leq 1$, we have

$$(41) \quad \lim_{x \rightarrow -\infty} \xi_x^{(\beta)} = \lim_{x \rightarrow -\infty} c_x \xi_1^{(\beta)} = 0.$$

Then by the theorem on functional Hellinger integrals in Chapter II,

$$(42) \quad P_t^{(\beta)} \mu \equiv P_{\pi_t(\beta)} \mu = \int_{-\infty}^t \frac{d(\mu, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2},$$

since, due to (37), $\xi_t^{(\beta)}$ satisfies condition (18), and

$$(43) \quad \|\xi_t^{(\beta)}\| \leq \|\xi_t^{(\beta)}\| + \|Q_t^{(\beta)} \xi_t^{(\beta)}\| \leq \|\xi_t^{(\beta)}\| + \|\xi_t^{(\beta)}\|.$$

From the additivity of Hellinger integrals, we have, for $s \equiv$ the smaller of t and r ,

$$\begin{aligned} C_t \int_{-\infty}^r \frac{d(\mu, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2} &= \int_{-\infty}^s \frac{d(\mu, \xi_x^{(\beta)}) d C_t \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2} \\ &\quad + \int_s^r \frac{d(\mu, \xi_x^{(\beta)}) d C_t \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2}. \end{aligned}$$

Clearly, for $r \leq t$, $C_t P_r^{(\beta)} \mu = P_r^{(\beta)} \mu$, as $C_t \xi_x^{(\beta)} = \xi_x^{(\beta)}$ when $x \leq r$. However, if $t \leq r$, the second integral $\int_s^r$ becomes $\int_t^r$ and is certainly zero, because, $C_t \xi_x^{(\beta)} = \xi_t^{(\beta)}$, $t \leq x \leq r$.

Thus in any case the following holds:

$$(44) \quad C_t P_r^{(\beta)} = P_s^{(\beta)}, \quad s \equiv \text{the smaller of } t, r.$$

Applying this result to $\xi_r^{(\alpha)}$ we find

$$C_t (P_r^{(\beta)} \xi_r^{(\alpha)}) = C_t (P_r^{(\beta)} C_r \xi_r^{(\alpha)}) = (C_t P_r^{(\beta)}) \xi_r^{(\alpha)} = P_s^{(\beta)} \xi_r^{(\alpha)}$$

$$= P_s^{(\beta)} \zeta_s^{(\alpha)},$$

so that $P_r^{(\beta)} \zeta_r^{(\alpha)}$ satisfies (37). But by (39) the spaces $\mathcal{H}_t^{(\beta)}$, $\beta < \alpha$, are orthogonal, and hence their projections are orthogonal. Consequently,

$$Q_t^{(\alpha)} \equiv P_{\mathcal{H}_t^{(\alpha)}} = \sum_{\beta < \alpha} P_{\mathcal{H}_t^{(\beta)}}, \quad Q_t^{(\alpha)} \zeta_t^{(\alpha)} = \sum_{\beta < \alpha} P_{\mathcal{H}_t^{(\beta)}} \zeta_t^{(\alpha)}.$$

It follows that $Q_t^{(\alpha)} \zeta_t^{(\alpha)}$ has property (37). Of course $\zeta_t^{(\alpha)}$ fulfills (37) from its definition (36) and the nature of C_t . This fact in conjunction with the proved property of $Q_t^{(\alpha)} \zeta_t^{(\alpha)}$ enables us to conclude at once that

$$\zeta_t^{(\alpha)} = \zeta_t^{(\alpha)} - Q_t^{(\alpha)} \zeta_t^{(\alpha)}$$

satisfies condition (37) also. Therefore, by the principle of transfinite induction, (37) holds for every system $\zeta_t^{(\alpha)}$.

As in (40) the condition (37) implies that every $\zeta_t^{(\alpha)}$ is strongly continuous in t . Further, we wish to show that $\Delta \zeta_t^{(\alpha)}$ is in $\mathcal{H}(\Omega)$ and

$$\Omega \Delta \zeta_t^{(\alpha)} = \int_{\Delta} t \, d \zeta_t^{(\alpha)}.$$

Because of the simple equalities

$$(45) \quad \begin{aligned} E_t \Delta \zeta_t^{(\alpha)} &= E_t \Delta (C \zeta_t^{(\alpha)}) = E_t C \Delta \zeta_t^{(\alpha)} = C_t \Delta \zeta_t^{(\alpha)}, \\ C_t \Delta \zeta_t^{(\alpha)} &= 0 \text{ for } t \leq \underline{\Delta}, \quad C_t \Delta \zeta_t^{(\alpha)} = \Delta \zeta_t^{(\alpha)} \text{ for } t \geq \overline{\Delta}, \end{aligned}$$

$\underline{\Delta}, \overline{\Delta}$ being the left and right end-points of Δ , we see that

$$\delta (C_t \Delta \zeta_t^{(\alpha)}) = 0$$

for δ in $(-\infty, \underline{\Delta})$ or $[\overline{\Delta}, +\infty)$. Similarly, for any δ in Δ ,

$$(46) \quad (E_t \Delta \zeta_t^{(\alpha)}) = \delta (C_t \Delta \zeta_t^{(\alpha)}) = (\zeta_{\overline{\delta}}^{(\alpha)} - \zeta_{\underline{\delta}}^{(\alpha)})$$

$$(46) \quad -(\xi_{\underline{s}}^{(\alpha)} - \xi_{\underline{t}}^{(\alpha)}) = \delta \xi_t^{(\alpha)},$$

on account of condition (37). It follows that

$$\int_{-\infty}^{+\infty} t^2 d(E_t \Delta \xi_t^{(\alpha)}, \Delta \xi_t^{(\alpha)}) = \int_{\Delta} t^2 d(C_t \Delta \xi_t^{(\alpha)}, \Delta \xi_t^{(\alpha)}) < +\infty,$$

and hence $\Delta \xi_t^{(\alpha)}$ lies in $\mathcal{M}(\Omega)$. Moreover,

$$\Omega \Delta \xi_t^{(\alpha)} = \int_{-\infty}^{+\infty} t dE_t \Delta \xi_t^{(\alpha)} = \int_{\Delta} t dC_t \Delta \xi_t^{(\alpha)} = \int_{\Delta} t d\xi_t^{(\alpha)},$$

again by (45), (46).

Let Q_t be the projection of $\sum_{\beta < \alpha} \mathcal{N}_t^{(\beta)}$. As $\xi_s^{(0)} = C_s \xi_s^{(0)}$, and $C_s \xi_s^{(0)}$ is in $C_s \mathcal{M}$ which is in $C_t \mathcal{M}$ for $s \leq t$, we see that $\xi_s^{(0)}$ must be in $C_t \mathcal{M}$, and hence $\mathcal{N}_t^{(0)}$ is in $C_t \mathcal{M}$. For the purpose of induction assume $\mathcal{N}_t^{(\beta)}$ in $C_t \mathcal{M}$ for every $\beta < \alpha$. Evidently then $\sum_{\beta < \alpha} \mathcal{N}_s^{(\beta)}$ is in $C_t \mathcal{M}$ and $Q_s^{(\alpha)} \mu$ lies in $C_t \mathcal{M}$, $s \leq t$. So,

$$\xi_s^{(\alpha)} = C_s \xi_s^{(\alpha)} - Q_s^{(\alpha)} \xi_s^{(\alpha)}$$

must lie in $C_t \mathcal{M}$ for $s \leq t$. But the totality of vectors $\xi_s^{(\alpha)}$ ($s \leq t$) is $\mathcal{N}_t^{(\alpha)}$ so that $\mathcal{N}_t^{(\alpha)}$ belongs to $C_t \mathcal{M}$. Thus, every $\mathcal{N}_t^{(\alpha)}$ lies in $C_t \mathcal{M}$, and so does $\sum_{\beta < \alpha} \mathcal{N}_t^{(\beta)}$. Then we can write $Q_t \leq C_t$. From definition (35),

$$C_t \xi_t^{(\alpha)} = \xi_t^{(\alpha)} = \xi_t^{(\alpha)} + Q_t^{(\alpha)} \xi_t^{(\alpha)}.$$

Now $\xi_t^{(\alpha)}$ lies in $\mathcal{N}_t^{(\alpha)}$ and hence in

$$\sum_{\beta < \alpha} \mathcal{N}_t^{(\beta)} = Q_t \mathcal{M}.$$

Also $Q_t \mathcal{M}$ contains $Q_t^{(\alpha)} \mathcal{M}$ as $Q_t^{(\alpha)}$ is the projection of $\sum_{\beta < \alpha} \mathcal{N}_t^{(\beta)}$. It follows that $\xi_t^{(\alpha)} = C_t \xi_t^{(\alpha)}$ is in $Q_t \mathcal{M}$. We recall that those basic vectors $\xi^{(\delta)}$ not in $[\xi^{(\alpha)} | \alpha]$ are orthogonal to C_t and so

$$C_t \mathcal{M} = C_t \left[\left\{ \zeta^{(\alpha)} \mid \alpha \right\} \right]_* \subset Q_t \mathcal{M}.$$

Consequently $C_t \leq Q_t$. This result with the one obtained above shows

$$(47) \quad Q_t = C_t.$$

By an argument parallel to that for (39) we infer that the spaces $\mathcal{M}_t^{(\alpha)}$ are mutually orthogonal, due to the definition of $\zeta_t^{(\alpha)}$ and condition (37). Since $C_t = Q_t$ which is the projection of $\sum \mathcal{M}_t^{(\alpha)}$,

$$(48) \quad C_t = \sum P_t^{(\alpha)}, \quad P_t^{(\alpha)} \equiv P_{\mathcal{M}_t^{(\alpha)}},$$

where the series converges strongly. This equality shows

$$(49) \quad Z \leq P_{t''}^{(\alpha)} - P_{t'}^{(\alpha)} \leq C_{t''} - C_{t'}, \quad t' \leq t'',$$

by virtue of (44). Now from (43) the vectors $\zeta_t^{(\alpha)}$ are uniformly bounded in t . Because of the weak compactness of $\mathcal{M}$ we can choose a denumerable sequence t_m such that $t_m \rightarrow +\infty$ and $\zeta_{t_m}^{(\alpha)}$ converges weakly to a limit $\zeta^{(\alpha)}$ as $m \rightarrow +\infty$. Consequently, for every μ of $\mathcal{M}$

$$\begin{aligned} (\zeta_r^{(\alpha)}, \mu) &= \lim_{m \rightarrow +\infty} (C_r \zeta_{t_m}^{(\alpha)}, \mu) = \lim_{m \rightarrow +\infty} (\zeta_{t_m}^{(\alpha)}, C_r \mu) \\ &= (\zeta^{(\alpha)}, C_r \mu) = (C_r \zeta^{(\alpha)}, \mu). \end{aligned}$$

Thus, $C_r \zeta^{(\alpha)} = \zeta_r^{(\alpha)}$. Then (49) gives

$$\begin{aligned} \|\zeta_{t''}^{(\alpha)} - \zeta_{t'}^{(\alpha)}\| &= \|(P_{t''}^{(\alpha)} - P_{t'}^{(\alpha)}) \zeta^{(\alpha)}\| \\ &\leq \|(C_{t''} - C_{t'}) \zeta^{(\alpha)}\| \leq \epsilon \end{aligned}$$

when t', t'' are sufficiently large, because

$$C_t \rightarrow C, \quad t \rightarrow +\infty, \quad C \equiv I - D.$$

It follows that $\zeta_t^{(\alpha)}$ converges strongly to $\zeta^{(\alpha)}$ as $t \rightarrow +\infty$. Hereafter we shall use only those vectors $\zeta_t^{(\alpha)}$ not identically zero in t . This means that $\zeta_t^{(\alpha)}$ cannot be a constant vector in $(-\infty, +\infty)$, for, if constant it would be zero from (41). We can then normalize $\zeta^{(\alpha)}$ so that $\|\zeta^{(\alpha)}\| = 1$.

With Hellinger we call a one-parameter family of vectors ζ_t a characteristic differential solution of Ω in case $\|\zeta_t\|$ is continuous and

$$\Omega \Delta \zeta_t = \int_{\Delta} t \, d\zeta_t.$$

Then we may formulate the results just obtained as follows:

THEOREM 8. For a self-adjoint operator Ω with C different from Z there exists a system of mutually orthogonal characteristic differential solutions $\{\zeta_t^{(\beta)} | \beta\}$ such that the linear closed extension of $\{\zeta_x^{(\beta)} | \beta\}$ coincides with $C_x \mathcal{M}$, $\zeta_t^{(\beta)}$ is strongly continuous, not constant on $(-\infty, +\infty)$, and

$$C_t \zeta_r^{(\beta)} = \zeta_s^{(\beta)}, \quad s \equiv \text{the smaller of } r, t.$$

Moreover,¹ $\zeta_t^{(\beta)}$ has a strong limit $\zeta^{(\beta)}$ as $t \rightarrow +\infty$, and

$$C_t \zeta^{(\beta)} = \zeta_t^{(\beta)}, \quad \|\zeta_t^{(\beta)}\| \leq 1, \quad \|\zeta^{(\beta)}\| = 1.$$

In Chapter IV we have shown

$$D_t = \sum_{r < t} \nabla_r.$$

Consider the non-vanishing ∇ 's. Take an orthonormal basis of $\nabla_r \mathcal{M}$ for each r . Then well order such vectors for all r 's and obtain $\{\zeta^{(\alpha)} | \alpha\}$. Let r_α be the characteristic value of $\zeta^{(\alpha)}$, since all these vectors are characteristic for Ω . Then vectors

¹This part was added after the author saw the book of Stone, while in New Haven. In a more geometrical form Stone gave the result for Hilbert space for the first time.

in the orthonormal basis of $\nabla_r \mathcal{M}$ are ordered by those α 's with $r_\alpha = r$, as vectors in $\nabla_r \mathcal{M}$ have the same characteristic value r . Conversely, if $r_\alpha = r$ the vector $\zeta^{(\alpha)}$ is in $\nabla_r \mathcal{M}$. It follows that the vectors $\zeta^{(\alpha)}$ with $r_\alpha = r$ coincide with the initial orthonormal basis of $\nabla_r \mathcal{M}$, and so

$$(50) \quad \nabla_r \mu = \sum_{\alpha, r_\alpha=r} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)},$$

where the series converges strongly because of its projection character. Consequently,

$$\begin{aligned} (D_t \mu, \mu) &= \mathbb{E} \sum_{r < t}^{\sigma} (\nabla_r \mu, \mu) = \mathbb{E} \sum_{r < t}^{\sigma} \sum_{\alpha, r_\alpha=r} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu) \\ &= \mathbb{E} \sum_{r < t}^{\sigma} \mathbb{E} \sum_{\alpha, r_\alpha=r}^{\tau} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu) \\ (51) \quad &= \mathbb{E} \mathbb{E} \sum_{r < t}^{\sigma} \sum_{\alpha, r_\alpha=r}^{\tau} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu) \\ &= \mathbb{E} \sum_{\alpha, r_\alpha < t}^{\tau} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu) \\ &= \sum_{\alpha, r_\alpha < t} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu). \end{aligned}$$

Now

$$(52) \quad \left\| D_t \mu - \sum_{\alpha, r_\alpha < t}^{\tau} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)} \right\|^2 = \|D_t \mu\|^2 + \left\| \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)} \right\|^2 - (D_t \mu, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}) - (\sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}, D_t \mu).$$

But for $R^c \equiv (r_\alpha | \alpha^c)$

$$\begin{aligned} (D_t \mu, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}) &= (\sum_r^{R^c} \nabla_r \mu, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}) = (\sum_r^{R^c} \sum_{\alpha, r_\alpha=r} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}) \\ &= (\sum_{\alpha, r_\alpha \in R^c} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}) = (\sum_{\alpha, r_\alpha \in R^c} \zeta^{(\alpha)}, \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}). \end{aligned}$$

Similarly,

$$(\sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)}, D_t \mu) = \left\| \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)} \right\|^2$$

Then (52) reduces to

$$(53) \quad \left\| D_t \mu - \sum_{\alpha, r_\alpha < t}^{\tau} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)} \right\|^2 = \|D_t \mu\|^2 - \left\| \sum_{\alpha, r_\alpha < t}^{\tau} \zeta^{(\alpha)} \right\|^2.$$

From the equality

$$\begin{aligned} \left\| \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)} \right\|^2 &= \left(\sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha}, \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} \right) = \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} ((\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}, \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha}) \\ &= \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} ((\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}, (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}) \\ &= \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} (\mu, \zeta^{(\alpha)}) (\zeta^{(\alpha)}, \mu), \end{aligned}$$

we find, with the aid of (51) and (53),

$$\frac{L}{\tau} \left\| D_t \mu - \sum_{\alpha^r \alpha < t} \frac{\tau}{\alpha^r \alpha} \right\|^2 = 0.$$

Thus

$$(54) \quad D_t \mu = \sum_{\alpha^r \alpha < t} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)}.$$

Further, in similar manner, we derive

$$(55) \quad D \mu = \sum_{\alpha} (\mu, \zeta^{(\alpha)}) \zeta^{(\alpha)},$$

making use of the formula

$$D = \sum_r \nabla_r.$$

Let $F(t)$ be a Borel measurable function for which

$$(56) \quad J \equiv \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \zeta, \zeta) < +\infty.$$

Then

$$\begin{aligned} J_c &\equiv \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_{ct} \zeta, \zeta) = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(C_t \zeta, \zeta) < +\infty, \\ (57) \quad J_d &\equiv \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_{dt} \zeta, \zeta) = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(D_t \zeta, \zeta) < +\infty. \end{aligned}$$

According to Theorem 4, the functional integrals

$$\begin{aligned} (58) \quad \gamma_c &\equiv \text{LS} \int_{-\infty}^{+\infty} F(t) dE_{ct} \zeta = \text{LS} \int_{-\infty}^{+\infty} F(t) dC_t \zeta, \\ \gamma_d &\equiv \text{LS} \int_{-\infty}^{+\infty} F(t) dE_{dt} \zeta = \text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \zeta \end{aligned}$$

exist as strong limits, and

$$(59) \quad \|\eta_c\|^2 = J_c, \quad \|\eta_d\|^2 = J_d.$$

From the fact that $(D_t \xi, \xi)$ is the jump function of $(E_t \xi, \xi)$ we deduce at once

$$\text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(D_t \xi, \xi) = \sum_r |F(r)|^2 (\nabla_r \xi, \xi).$$

By an argument similar to that of (51) we see that

$$\sum_r |F(r)|^2 (\nabla_r \xi, \xi) = \sum_{\alpha} |F(r_{\alpha})|^2 (\xi, \xi^{(\alpha)}) (\xi^{(\alpha)}, \xi),$$

which gives

$$(60) \quad \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(D_t \xi, \xi) = \sum_{\alpha} |F(r_{\alpha})|^2 (\xi, \xi^{(\alpha)}) (\xi^{(\alpha)}, \xi).$$

We also readily verify

$$(\text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \xi, \sum_{\alpha} F(r_{\alpha}) (\xi, \xi^{(\alpha)}) \xi^{(\alpha)}) = \left\| \sum_{\alpha} \frac{\sigma}{\alpha} \right\|^2,$$

so that

$$\begin{aligned} \left\| \text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \xi - \sum_{\alpha} \frac{\sigma}{\alpha} F(r_{\alpha}) (\xi, \xi^{(\alpha)}) \xi^{(\alpha)} \right\|^2 &= \left\| \text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \xi \right\|^2 \\ &- \left\| \sum_{\alpha} \frac{\sigma}{\alpha} \right\|^2 = \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(D_t \xi, \xi) - \sum_{\alpha} |F(r_{\alpha})|^2 (\xi, \xi^{(\alpha)}) (\xi^{(\alpha)}, \xi). \end{aligned}$$

This, by virtue of (60), shows that

$$(61) \quad \text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \xi = \sum_{\alpha} F(r_{\alpha}) (\xi, \xi^{(\alpha)}) \xi^{(\alpha)},$$

where the series converges strongly.

We now return to C_t again. From (49) we conclude that $P_t(\beta)$ converges strongly to $P(\beta)$ as $t \rightarrow +\infty$, and

$$(62) \quad \begin{aligned} P_t(\beta) \xi^{(\beta)} &\rightarrow \xi^{(\beta)}, \quad t \rightarrow +\infty, \quad P_t(\beta) P(\beta) = P_t(\beta), \\ P(\beta) \mu &= \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2}. \end{aligned}$$

Because of the orthogonality relation, we see that $P(\beta_1)P_t(\beta_2) = Z$, $\beta_1 \neq \beta_2$, from which follows $C_t P(\beta) = P_t(\beta)$, in consequence of (48). By (48) and the monotone increasing property of C_t and $P_t(\beta)$,

$$\begin{aligned}
 (C\mu, \mu) &= \overline{E}_t(C_t\mu, \mu) = \overline{E}_t \sum_{\beta} (P_t(\beta)\mu, \mu) \\
 (63) \quad &= \overline{E}_t \overline{E}_{\sigma} \sum_{\beta} (P_t(\beta)\mu, \mu) = \overline{E}_{\sigma} \overline{E}_t \sum_{\beta} (P_t(\beta)\mu, \mu) \\
 &= \sum_{\beta} (P(\beta)\mu, \mu).
 \end{aligned}$$

Polarizing this equality we find $\sum P(\beta)$ converging weakly to C . Because of the orthogonal property of the P 's we verify without difficulty

$$\|C\mu - \sum_{\beta} P(\beta)\mu\|^2 = \|C\mu\|^2 - \sum_{\beta} \|P(\beta)\mu\|^2,$$

which, on account of (63), yields

$$(64) \quad C = \sum_{\beta} P(\beta),$$

the series being strongly convergent. By virtue of (62), (48) and (42), we can also write

$$\begin{aligned}
 (65) \quad C\mu &= \sum_{\beta} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x(\beta)) d\xi_x(\beta)}{d\|\xi_x(\beta)\|^2}, \\
 C_t\mu &= \sum_{\beta} \int_{-\infty}^t \frac{d(\mu, \xi_x(\beta)) d\xi_x(\beta)}{d\|\xi_x(\beta)\|^2}.
 \end{aligned}$$

Under condition (56) for a Borel measurable function $F(t)$ we have (57), and

$$\begin{aligned}
 (66) \quad \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(C_t \xi, \xi) &= \sum_{\beta} \text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 \cdot \\
 &\quad d \int_{-\infty}^t \frac{d(\xi, \xi_x(\beta)) d(\xi_x(\beta), \xi)}{d\|\xi_x(\beta)\|^2},
 \end{aligned}$$

in view of (65) and Theorem 3. From (49) the Lebesgue-Stieltjes integral of $|F(t)|^2$ relative to $(P_t^{(\beta)}\{\}, \{\})$ exists, and hence the functional integral

$$(67) \quad LS \int_{-\infty}^{+\infty} F(t) dP_t^{(\beta)}\{\}$$

exists as a strong limit by Theorem 4. Let

$$\{\}^* \equiv LS \int_{-\infty}^{+\infty} F(t) dC_t\{\}, \quad \{\}^{(\beta)} \equiv LS \int_{-\infty}^{+\infty} F(t) dP_t^{(\beta)}\{\}.$$

Then by Theorem 4

$$(68) \quad \|\{\}^{(\beta)}\|^2 = LS \int_{-\infty}^{+\infty} |F(t)|^2 d(P_t^{(\beta)}\{\}, \{\}).$$

From the equality

$$\begin{aligned} (P_t^{(\beta)}\{\}, \{\}^*) &= (\{\}, P_t^{(\beta)}\{\}^*) = (\{\}, \int_{-\infty}^{+\infty} F(x) dC_x P_t^{(\beta)}\{\}) \\ &= (\{\}, \int_{-\infty}^t F(x) dP_x^{(\beta)}\{\}) \end{aligned}$$

derived from (44) we find

$$\begin{aligned} (\{\}^{(\beta)}, \{\}^*) &= \int_{-\infty}^{+\infty} F(t) d(P_t^{(\beta)}\{\}, \{\}^*) = \int_{-\infty}^{+\infty} F(t) d(\{\}, \int_{-\infty}^t F(x) dP_x^{(\beta)}\{\}) \\ &= \int_{-\infty}^{+\infty} F(t) d(\{\}, P_t^{(\beta)}\{\}) \int_{-\infty}^{+\infty} F(x) dP_x^{(\beta)}\{\} \\ &= \int_{-\infty}^{+\infty} F(t) d(P_t^{(\beta)}\{\}, \int_{-\infty}^{+\infty} F(x) dP_x^{(\beta)}\{\}) \\ &= \int_{-\infty}^{+\infty} F(t) d(P_t^{(\beta)}\{\}, \{\}^{(\beta)}) = \|\{\}^{(\beta)}\|^2 \end{aligned}$$

in accordance with Theorem 4, the integrals being of the Lebesgue-Stieltjes type. Thus,

$$\begin{aligned} \|\{\}^* - \frac{\sigma}{\beta} \{\}^{(\beta)}\|^2 &= \|\{\}^*\|^2 + \|\frac{\sigma}{\beta} \{\}^{(\beta)}\|^2 - (\frac{\sigma}{\beta} \{\}^{(\beta)}, \{\}^*) - (\{\}^*, \frac{\sigma}{\beta} \{\}^{(\beta)}) \\ &= \|\{\}^*\|^2 + \frac{\sigma}{\beta} \|\{\}^{(\beta)}\|^2 - \frac{\sigma}{\beta} \|\{\}^{(\beta)}\|^2 - \frac{\sigma}{\beta} \|\{\}^{(\beta)}\|^2 = \|\{\}^*\|^2 - \frac{\sigma}{\beta} \|\{\}^{(\beta)}\|^2, \end{aligned}$$

which, with the aid of (66) and (68), gives

$$\begin{aligned}
 \text{LS} \int_{-\infty}^{+\infty} F(t) dC_t \{ &= \{ \}^* = \sum_{\beta} \{ \}^{(\beta)} \\
 (69) \qquad \qquad \qquad &= \sum_{\beta} \text{LS} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\{, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2},
 \end{aligned}$$

where the series converges strongly.

However, $I = D + C$, and

$$\text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \{ = \text{LS} \int_{-\infty}^{+\infty} F(t) dD_t \{ + \text{LS} \int_{-\infty}^{+\infty} F(t) dC_t \{ ,$$

so that combining the results in (55), (65), (61), (69), we finally reach the next theorem of characteristic expansions.

THEOREM 9. Relative to a self-adjoint operator Ω in a non-zero space, every function μ of $\mathcal{M}$ can be expanded into a strongly convergent series:

$$\mu = \sum_{\alpha} (\mu, \xi^{(\alpha)}) \xi^{(\alpha)} + \sum_{\beta} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_t^{(\beta)}) d \xi_t^{(\beta)}}{d \|\xi_t^{(\beta)}\|^2},$$

where $\{\xi^{(\alpha)} | \alpha\}$ is an orthonormal system of characteristic functions of Ω complete for $D\mathcal{M}$, and $\{\xi_t^{(\beta)} | \beta\}$ is an orthonormal system of characteristic differential solutions of Ω complete for $C\mathcal{M}$. The two systems are also mutually orthogonal.

If a function $\{$ in $\mathcal{M}$ and a real Borel measurable function $F(t)$ satisfy the relation

$$\text{LS} \int_{-\infty}^{+\infty} |F(t)|^2 d(E_t \{, \{) < +\infty,$$

then

$$\begin{aligned}
 \text{LS} \int_{-\infty}^{+\infty} F(t) dE_t \{ &= \sum_{\alpha} F(r_{\alpha}) (\{, \xi^{(\alpha)}) \xi^{(\alpha)} \\
 &+ \sum_{\beta} \text{LS} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\{, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2},
 \end{aligned}$$

where r_{α} is the characteristic value for $\xi^{(\alpha)}$ with respect to Ω , and the two series converge strongly.

We observe that the resolutions of the identity with respect to

$$\Omega_b \equiv \int_{-\infty}^{+\infty} t \, dB_t, \quad \Omega_a \equiv \int_{-\infty}^{+\infty} t \, dA_t,$$

are given by

$$E_{bt} = B_t \text{ for } t \leq 0, \quad E_{bt} = B_t + (I - B) \text{ for } t > 0, \quad B \equiv \lim_{t \rightarrow +\infty} B_t,$$

$$E_{at} = A_t \text{ for } t \leq 0, \quad E_{at} = A_t + (I - A) \text{ for } t > 0, \quad A \equiv \lim_{t \rightarrow +\infty} A_t.$$

Then the continuous part of E_{bt} is B_t . Applying Theorem 8 to Ω_b , we see that there exists a system $\{\xi_t^{(\beta)} | \beta\}$ of mutually orthogonal characteristic differential solutions having all the properties and relations mentioned in that theorem, only that these relations now hold for B_t instead of C_t . Moreover, since $(B_t \xi, \mu)$ has a derivative almost everywhere zero.

$$(\xi_t^{(\beta)}, \mu) = (B_t \xi^{(\beta)}, \mu),$$

$$\|\xi_t^{(\beta)}\|^2 = (B_t \xi^{(\beta)}, B_t \xi^{(\beta)}) = (B_t \xi^{(\beta)}, \xi^{(\beta)})$$

both have derivatives almost everywhere zero. Since $\|\xi_t^{(\beta)}\|$ has zero derivative at every point where $\|\xi_t^{(\beta)}\|^2$ has zero derivative with the possible exception of only one point, $\|\xi_t^{(\beta)}\|$ has therefore zero derivative almost everywhere. Indeed, we can also apply (65) and (69), which in this case become expansions for $B\mu$, and

$$LS \int_{-\infty}^{+\infty} F(t) \, dB_t \Big\}.$$

Again applying Theorem 8 and expansions (65), (69), we obtain similar results for Ω_a . However, each characteristic differential solution $\xi_t^{(\alpha)}$ has the further property that it is strongly absolutely continuous in t , and

$$(\zeta_t^{(\alpha)}, \mu) = (A_t \zeta^{(\alpha)}, \mu), \quad \|\zeta_t^{(\alpha)}\|^2 = (A_t \zeta^{(\alpha)}, \zeta^{(\alpha)})$$

are indefinite integrals of their derivatives, since A_t is strongly absolutely continuous in t . Then by a theorem on absolutely continuous functions,¹ it can be readily proved that $\|\zeta_t^{(\alpha)}\|$ is absolutely continuous. Finally, we recall that $I = D + B + A$, and $E_t = D_t + B_t + A_t$.

Summing up, we have now the principal theorem on characteristic expansions.

THEOREM 10. For a self-adjoint operator Ω in a non-zero space there exist an orthonormal system of characteristic functions $\{\zeta^{(\delta)} | \delta\}$ with characteristic values η_δ , and two systems of mutually orthogonal characteristic differential solutions $\{\zeta_t^{(\beta)} | \beta\}$, $\{\zeta_t^{(\alpha)} | \alpha\}$ having the following properties:

- (a) These three systems are complete for $D\mathcal{M}$, $B\mathcal{M}$, $A\mathcal{M}$ respectively.
- (b) The linear closed extensions of $\{\zeta_t^{(\beta)} | \beta\}$, $\{\zeta_t^{(\alpha)} | \alpha\}$ coincide respectively with $B_t\mathcal{M}$, $A_t\mathcal{M}$.
- (c) As functions of t , $\zeta_t^{(\beta)}$, $\zeta_t^{(\alpha)}$ are strongly continuous, not constant, have strong limits $\zeta^{(\beta)}$, $\zeta^{(\alpha)}$ as $t \rightarrow +\infty$, and
- $$B_t \zeta_r^{(\beta)} = \zeta_s^{(\beta)}, \quad A_t \zeta_r^{(\alpha)} = \zeta_s^{(\alpha)}, \quad s \equiv \text{the smaller of } r, t,$$
- $$B_t \zeta^{(\beta)} = \zeta_t^{(\beta)}, \quad A_t \zeta^{(\alpha)} = \zeta_t^{(\alpha)}, \quad \|\zeta_t^{(\beta)}\| \leq 1, \quad \|\zeta_t^{(\alpha)}\| \leq 1,$$
- $$\|\zeta^{(\beta)}\| = 1 = \|\zeta^{(\alpha)}\|.$$
- (d) The numerical functions $\|\zeta_t^{(\beta)}\|$, $(\zeta_t^{(\beta)}, \mu)$ have derivatives almost everywhere zero.

- (e) $\zeta_t^{(\alpha)}$ is strongly absolutely continuous with $\|\zeta_t^{(\alpha)}\|$,

¹Hahn, Theorie der reellen Funktionen, p. 527.

$(\xi_t^{(\alpha)}, \mu)$ equal to the indefinite integrals of their derivatives.

With respect to these systems every function μ can be expanded into a strongly convergent series, every inner product (μ, η) into an absolutely convergent series:

$$\mu = \sum_{\delta} (\mu, \xi^{(\delta)}) \xi^{(\delta)} + \sum_{\beta} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x^{(\beta)}) d\xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2} + \sum_{\alpha} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x^{(\alpha)}) d\xi_x^{(\alpha)}}{d \|\xi_x^{(\alpha)}\|^2},$$

$$(\mu, \eta) = \sum_{\delta} (\mu, \xi^{(\delta)}) (\xi^{(\delta)}, \eta) + \sum_{\beta} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x^{(\beta)}) d(\xi_x^{(\beta)}, \eta)}{d \|\xi_x^{(\beta)}\|^2} + \sum_{\alpha} \int_{-\infty}^{+\infty} \frac{d(\mu, \xi_x^{(\alpha)}) d(\xi_x^{(\alpha)}, \eta)}{d \|\xi_x^{(\alpha)}\|^2}.$$

If a function ξ of the class $\mathcal{M}$ and a real Borel measurable function $F(t)$ have a finite Lebesgue-Stieltjes integral of $|F(t)|^2$ relative to $(E_t \xi, \xi)$, then

$$\begin{aligned} \int_{-\infty}^{+\infty} F(t) dE_t \xi &= \sum_{\delta} F(r_{\delta}) (\xi, \xi^{(\delta)}) \xi^{(\delta)} \\ &+ \sum_{\beta} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\beta)}) d\xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2} \\ &+ \sum_{\alpha} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\alpha)}) d\xi_x^{(\alpha)}}{d \|\xi_x^{(\alpha)}\|^2}, \\ \left\| \int_{-\infty}^{+\infty} F(t) dE_t \xi \right\|^2 &= \sum_{\delta} |F(r_{\delta})|^2 (\xi, \xi^{(\delta)}) (\xi^{(\delta)}, \xi) \\ &+ \sum_{\beta} \int_{-\infty}^{+\infty} |F(t)|^2 d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\beta)}) d(\xi_x^{(\beta)}, \xi)}{d \|\xi_x^{(\beta)}\|^2} \\ &+ \sum_{\alpha} \int_{-\infty}^{+\infty} |F(t)|^2 d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\alpha)}) d(\xi_x^{(\alpha)}, \xi)}{d \|\xi_x^{(\alpha)}\|^2}, \\ \left(\int_{-\infty}^{+\infty} F(t) dE_t \xi, \eta \right) &= \sum_{\delta} F(r_{\delta}) (\xi, \xi^{(\delta)}) (\xi^{(\delta)}, \eta) \\ &+ \sum_{\beta} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\beta)}) d(\xi_x^{(\beta)}, \eta)}{d \|\xi_x^{(\beta)}\|^2} \\ &+ \sum_{\alpha} \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\alpha)}) d(\xi_x^{(\alpha)}, \eta)}{d \|\xi_x^{(\alpha)}\|^2}, \end{aligned}$$

where the integrals are taken in the sense of Lebesgue-Stieltjes, the functional series converge strongly, and the numerical series converge absolutely.

In particular, when ξ , and the continuous real $F(t)$ have a finite Stieltjes integral of $|F(t)|^2$ with respect to $(E_t \xi, \xi)$, the functional Stieltjes integrals of $F(t)$ with respect to $E_t \xi$, to $P_t^{(\beta)} \xi$ and to $P_t^{(\alpha)} \xi$ exist as strong limits by Theorem 5 and (67). Consequently, from the theorem on functional Hellinger integrals in Chapter II,

$$P_t^{(\beta)} \xi = \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2},$$

and hence

$$\int_{-\infty}^{+\infty} F(t) dP_t^{(\beta)} \xi = \int_{-\infty}^{+\infty} F(t) d \int_{-\infty}^t \frac{d(\xi, \xi_x^{(\beta)}) d \xi_x^{(\beta)}}{d \|\xi_x^{(\beta)}\|^2},$$

which reduces to the Hellinger integral

$$\int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t^{(\beta)}) d \xi_t^{(\beta)}}{d \|\xi_t^{(\beta)}\|^2},$$

by virtue of Theorem 7. We are thus led to the next theorem.

THEOREM 11. When a ξ and a continuous real function $F(t)$ have a finite Stieltjes integral of $|F(t)|^2$ with respect to $(E_t \xi, \xi)$, the following expansions hold:

$$\begin{aligned} \int_{-\infty}^{+\infty} F(t) dE_t \xi &= \sum_{\delta} F(r_{\delta})(\xi, \xi^{(\delta)}) \xi^{(\delta)} \\ &+ \sum_{\beta} \int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t^{(\beta)}) d \xi_t^{(\beta)}}{d \|\xi_t^{(\beta)}\|^2} \\ &+ \sum_{\alpha} \int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t^{(\alpha)}) d \xi_t^{(\alpha)}}{d \|\xi_t^{(\alpha)}\|^2}, \\ (S \int_{-\infty}^{+\infty} F(t) dE_t \xi, \gamma) &= \sum_{\delta} F(r_{\delta})(\xi, \xi^{(\delta)})(\xi^{(\delta)}, \gamma) + \end{aligned}$$

$$\begin{aligned}
& + \sum_{\beta} \int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t^{(\beta)}) d(\xi_t^{(\beta)}, \eta)}{d \|\xi_t^{(\beta)}\|^2} \\
& + \sum_{\alpha} \int_{-\infty}^{+\infty} F(t) \frac{d(\xi, \xi_t^{(\alpha)}) d(\xi_t^{(\alpha)}, \eta)}{d \|\xi_t^{(\alpha)}\|^2},
\end{aligned}$$

where the integrals in the expansions are of the Hellinger type.

In particular, for any function ξ in the independent range of Ω we have

$$\begin{aligned}
\Omega \xi &= \sum_{\delta} R_{\delta}(\xi, \xi_t^{(\delta)}) \xi_t^{(\delta)} + \sum_{\beta} \int_{-\infty}^{+\infty} t \frac{d(\xi, \xi_t^{(\beta)}) d \xi_t^{(\beta)}}{d \|\xi_t^{(\beta)}\|^2} + \sum_{\alpha} \int_{-\infty}^{+\infty} t \frac{d(\xi, \xi_t^{(\alpha)}) d \xi_t^{(\alpha)}}{d \|\xi_t^{(\alpha)}\|^2}, \\
(\Omega \xi, \eta) &= \sum_{\delta} R_{\delta}(\xi, \xi_t^{(\delta)}) (\xi_t^{(\delta)}, \eta) + \sum_{\beta} \int_{-\infty}^{+\infty} t \frac{d(\xi, \xi_t^{(\beta)}) d \xi_t^{(\beta)}, \eta)}{d \|\xi_t^{(\beta)}\|^2} + \sum_{\alpha} \int_{-\infty}^{+\infty} t \frac{d(\xi, \xi_t^{(\alpha)}) d \xi_t^{(\alpha)}, \eta)}{d \|\xi_t^{(\alpha)}\|^2}.
\end{aligned}$$

It should be noted that this theorem gives a strict generalization of the classical expansions by means of Hellinger integrals. Also it strengthens them since strongly convergent integrals are used. Further, it is sharper than the standard results as we have here a three-part decomposition.

APPENDIX

REFERENCES FROM E. H. MOORE'S GENERAL ANALYSIS

1. General reciprocal of a hermitian matrix

As before let $\mathcal{A}$ be the set of all real numbers, or of all complex numbers, or of all real quaternions, $\mathcal{P}$ be an unconditioned class of members p . A square matrix K on $\mathcal{P}\mathcal{P}$ to $\mathcal{A}$ is a function of two arguments each varying independently over $\mathcal{P}$, with the matricial element $K(p_1 p_2)$ at p_1 -row and p_2 -column. If the arguments vary only in a finite subset σ of $\mathcal{P}$, then $K(\sigma \sigma)$, having a finite number of rows and columns, is the principal minor of K on σ .

The matrix is hermitian if $\overline{K(p_1 p_2)} = K(p_2 p_1)$ for every pair $p_1 p_2$. For K on a finite range a determinant can be defined. This determinant is always real-valued. The matrix is said to be positive if the determinants of its principal minors are all positive. This is so if and only if the form

$$\sum_{p_1} \sum_{p_2} \{ (p_1) K(p_1 p_2) \} \overline{\{ (p_2) \}} \geq 0$$

for every $\{$. A hermitian matrix K on a general range is positive when its finite principal minors are positive.

It can be shown that every hermitian matrix K on a finite range has the canonical form¹

$$K = \sum_m \lambda_m \Delta_m,$$

¹cf. H. Weyl, Gruppentheorie und Quantenmechanik, 1931, p. 23.

where r_m is a real characteristic value of K and Δ_m the idempotent hermitian matrix associated with r_m . Then we call the following matrix the general reciprocal¹ of K :

$$K' \equiv \sum_m \frac{1}{r_m'} \Delta_m ;$$

the prime indicating that the fraction is taken to be zero when the denominator vanishes.

2. General limits

Suppose that $\mathcal{L}$ is an abstract class of elements χ among which holds a binary relation R . We call R transitive, if χ_1 is in the relation R to χ_3 whenever χ_1 is in R to χ_2 and χ_2 is in R to χ_3 . It has the composition property if for every two (not necessarily distinct) elements there exists another which is in the R relation to each of the initial elements. Then for such a relation R we say that a numerical (single-valued or otherwise) function α on $\mathcal{L}$ converges,² with respect to R , to a number a as limit, in case for every positive number ϵ there exists an χ_ϵ such that $|\alpha(\chi) - a|$ is at most ϵ for any χ in R to χ_ϵ . We write

$$\chi_1 R \chi_2, \quad \lim_{\chi R} \alpha(\chi) = a,$$

respectively for χ_1 in $R\chi_2$, $\alpha(\chi)$ converging to a (relative to R).

It is to be noted that all the finite subsets σ of a given set $\mathcal{P}$ form a class over which inclusion ($\sigma_1 \supset \sigma_2$) holds as a binary transitive relation having the composition property.

¹This definition is not the one originally given by Moore. In his work general reciprocals are defined and proved to exist for any rectangular matrix on finite ranges. However, for hermitian matrices the present definition is equivalent to that of Moore.

²See E. H. Moore and H. L. Smith, American Journal of Math. 44 (1922), pp. 102-21.

Limits defined as to the inclusion relation are frequently referred to as "limits as σ swells."

The upper and lower limits of a real function $\rho(\chi)$ are defined as follows. Let $\rho^*(\chi)$ be the least upper bound of the functional values $\rho(\chi_0)$ for $\chi_0 R \chi$. Then $\rho^*(\chi)$ is a monotone decreasing function of χ . The greatest lower bound of the values $\rho^*(\chi)$ is the upper limit $\overline{\lim} \rho(\chi)$ of $\rho(\chi)$. A similar definition applies to the lower limit.

As in the more familiar situations the Cauchy condition also holds for general limits. In order that a function $\alpha(\chi)$ shall converge to a finite number it is necessary and sufficient that for every positive ϵ there exists an χ_ϵ with

$$|\alpha(\chi_1) - \alpha(\chi_2)| \leq \epsilon$$

for $\chi_1 R \chi_\epsilon$, $\chi_2 R \chi_\epsilon$. Again, a function $\alpha(\chi p)$ containing a parameter p converges uniformly if and only if for every $\epsilon > 0$ there is an χ_ϵ , independent of p , such that

$$|\alpha(\chi_1 p) - \alpha(\chi_2 p)| \leq \epsilon$$

for any p , whenever $\chi_1 R \chi_\epsilon$ and $\chi_2 R \chi_\epsilon$.

3. Modular functions and the J-operation

Now consider a positive hermitian matrix ϵ on an unrestricted range $\mathcal{P}$. For a function ξ on $\mathcal{P}$ to $\mathcal{A}$ we define its modulus relative to ϵ as the least upper bound of $|\frac{\sigma}{\lambda} \xi(p) \overline{\alpha}(p)|$ for every subset σ and every function α on σ to $\mathcal{A}$ satisfying the condition $\sum_{\lambda_1}^{\sigma} \sum_{\lambda_2}^{\sigma} \alpha(p_1) \epsilon(p_1 p_2) \overline{\alpha}(p_2) \leq 1$. This modulus is denoted by $M \xi$. If $M \xi$ is finite ξ is said to be modular relative to ϵ . For any pair of numerical functions ξ, η we define

$J_{\sigma}\{\bar{\eta}$ to be $\sum_{p_1}^{\sigma} \sum_{p_2}^{\sigma} \{ (p_1) \gamma_{\sigma}(p_1 p_2) \bar{\eta}(p_2) \}$ where $\gamma_{\sigma}(\sigma\sigma)$ is the general reciprocal of $\epsilon(\sigma\sigma)$. When the limit $\lim_{\sigma} J_{\sigma}\{\bar{\eta}$ exists, its value is written as $J\{\bar{\eta}$. We call $\{$ accordant relative to ϵ in case $\sum_{p_1}^{\sigma} \{ (p) \alpha(p) = 0$ whenever $\sum_{p_1}^{\sigma} \sum_{p_2}^{\sigma} \alpha(p_1) \epsilon(p_1 p_2) \alpha(p_2) = 0$.

A function vanishes identically if and only if its modulus is zero. On a finite range $\{$ is modular when and only when $\{$ is accordant or, what is the same, when $\{$ is a left finite linear combination of the rows of ϵ . For an accordant $\{$ the positive function $J_{\sigma}\{\bar{\{}$ increases as σ swells. Moreover, in order that $\{$ shall be modular it is necessary and sufficient that $\{$ is accordant and that $J_{\sigma}\{\bar{\{}$ is bounded. If $\{, \gamma$ are modular, then the operation $J\{\bar{\gamma}$ exists. In fact, $(M\{\})^2 \equiv J\{\bar{\{}$. Every row $\epsilon(p\sigma)$ of ϵ is modular having $(\epsilon(p\sigma))^{1/2}$ as its modulus.

Given an arbitrary function $\{$ we form its modified function $\{\sigma$ (as to σ) thus: $\{\sigma \equiv J_{\sigma}\{\epsilon$. Then $\{\sigma$ is a left finite linear combination of the rows of ϵ and hence is modular relative to ϵ . For $\tau \supset \sigma$ we have $\{\sigma = (\{\sigma)_{\tau}$. Let γ be another arbitrary numerical function. Then

$$J_{\sigma}\{\bar{\gamma} = J\{\sigma\bar{\gamma}_{\sigma} = J_{\sigma}\{\bar{\gamma}_{\sigma} = J_{\sigma}\{\sigma\bar{\gamma}.$$

In case $\{$ is accordant on τ , $\{(\tau) = \{\tau(\tau)$, and $(\{\tau)_{\sigma} = \{\sigma$, σ being a subset of τ . If $\{$ is accordant relative to ϵ , the next equalities hold:

$$J_{\sigma}\{\bar{\gamma} = \lim_{\sigma} J_{\sigma}\{\bar{\gamma}_{\sigma} \equiv J\{\bar{\gamma}_{\sigma}, \quad J_{\sigma}\gamma\bar{\{} = J\gamma_{\sigma}\bar{\{}.$$

Finally, for any modular function μ we have

$$\lim_{\sigma} M(\mu - \mu_{\sigma}) = 0.$$

Let $\{\mu_\chi\}$ be an $\mathcal{L}$ R-system of modular functions. Suppose that they converge to a function ζ and that the upper limit of their moduli is finite. Then ζ is modular having a modulus not greater than the lower limit of the moduli. Furthermore,

$$\lim_{\chi} J \mu_\chi \overline{\mu} = J \zeta \overline{\mu}$$

for every modular function μ .

If $\{\mu_m\}$ is a sequence of modular functions with their moduli uniformly bounded, then there exists an infinite subsequence $\{\mu_{m_n}\}$ which converges to a modular function.

4. Hellinger integrals

Let β be a real function defined on an unrestricted set $\mathcal{P}$ of elements p , $\underline{b}$ be its finite greatest lower bound. In what follows we shall use t to designate numbers not greater than $\underline{b}$.

We call a function on $\mathcal{P}$ to $\mathcal{A}$ accordant (with β) in case $\zeta(p_1) = \zeta(p_2)$ whenever $\beta(p_1) = \beta(p_2)$. If $\zeta(p) = 0$ whenever $\beta(p) = t$, then ζ is said to be accordant (with β) at t .

Now introduce a matrix ϵ_t by the equation

$$\epsilon_t(p_1 p_2) \equiv \min. (\beta(p_1), \beta(p_2)) - t.$$

Then ϵ_t is a positive hermitian matrix. Consequently we have accordant, modular functions, and J-operation with respect to ϵ_t . For any pair of numerical functions ζ, η , define

$$J_{t\sigma} \zeta \overline{\eta} \equiv \sum_{p_1}^{\sigma} \sum_{p_2}^{\sigma} \zeta(p_1) \gamma_{t\sigma}(p_1 p_2) \overline{\eta}(p_2),$$

where $\gamma_{t\sigma}$ is the general reciprocal of $\epsilon_{t\sigma}$. We write J_t for the J-operation relative to ϵ_t . It may be noted that if ζ is accordant (with β) it is accordant with ϵ_t for every t . Moreover,

accordance at t and accordance with ϵ_t are equivalent conditions.

Consider a finite subset σ of $\mathcal{P}$. Let p_σ stand for an element in σ such that $\beta(p_\sigma)$ equals the minimum of $\beta(p)$ for all the p 's in σ . If ξ, η are both accordant, put

$$H_\sigma \xi \bar{\eta} \equiv \sum \frac{\Delta \xi \Delta \bar{\eta}}{\Delta \beta} = \sum_{n=1}^{s-1} \frac{[\xi(p_{n+1}) - \xi(p_n)] [\bar{\eta}(p_{n+1}) - \bar{\eta}(p_n)]}{\beta(p_{n+1}) - \beta(p_n)},$$

where $\sigma = [p_1 \cdots p_s]$ with $\beta(p_1) \leq \cdots \leq \beta(p_s)$, and the fraction is zero when its denominator vanishes.

If ξ, η are accordant at t (or accordant with ϵ_t) then

$$J_{t\sigma} \xi \bar{\eta} = \frac{\xi(p_\sigma) \bar{\eta}(p_\sigma)}{\beta(p_\sigma) - t} + H_\sigma \xi \bar{\eta}.$$

Suppose that ξ, η are modular relative to $\epsilon_{\underline{b}}$. Then

$$\int_{\sigma}^L \frac{\xi(p_\sigma) \bar{\eta}(p_\sigma)}{\beta(p_\sigma) - t} = 0,$$

$$\int_{\sigma}^L H_\sigma \xi \bar{\eta} = J_t \xi \bar{\eta}.$$

Let $\beta(x)$ be a real continuous monotone increasing function on $[a, b]$. If ξ is accordant (with β) and if $\int_{\sigma}^L H_\sigma \xi \bar{\xi}$ exists, then the Hellinger integral $\int_a^b d\xi d\bar{\xi}/d\beta$ exists and is equal to $\int_{\sigma}^L H_\sigma \xi \bar{\xi}$.

VITA

Yuan-Yung Tseng was born on September 13, 1904 in Nanki, Szechuan, China. He received his early education in public schools in Szechuan. In 1919 he entered the Middle School of Tsing Hua (Junior) College at Peking, where he was graduated in 1927. While at Tsing Hua he studied mathematics under Professors K. L. Hiong and C. F. Tsen. In October, 1927 he came to the University of Chicago, receiving the S.B. degree in August, 1928. The next year, under Professor L. M. Graves he wrote a thesis in the theory of Lebesgue-Stieltjes integration and received the S.M. degree in March, 1930. During the years 1927-1931 he was in constant residence at the University of Chicago, attending courses of Professors R. W. Barnard, W. Bartky, G. A. Bliss, L. E. Dickson, L. M. Graves, E. Hille, E. P. Lane, M. I. Logsdon, A. L. Lunn, E. H. Moore, F. D. Murnaghan, and H. E. Slaughter. After the preceding work was practically completed he went to Princeton to study under Professor J. von Neumann in the first semester of 1931-1932, and spent the second semester studying under Professor M. H. Stone at Yale. The author is greatly indebted to all of his instructors. He owes very much to Professors Hille and Ore, whose seminars he had the privilege to attend. To Professors Moore, Barnard, Graves, von Neumann, and Stone he wishes to express his sincere gratitude for their encouragement and interest in his work. In connection with the present thesis, the author is especially grateful to Professors Moore and Barnard, under whom this research was carried out.

